A SPLITTING LEMMA FOR BIHOLOMORPHIC MAPS ON CONTINUOUSLY VARYING DOMAINS

by

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Contents

CHAPTER 1

Introduction

In this thesis we construct compositional splittings of biholomorphic maps close to the identity (on domains varying with a parameter) and show that the maps obtained from this construction depend continuously on the parameter. The main result of this thesis will be an application of said construction in an important situation. In the first section of this chapter we will motivate the main result and explain what it is needed for. In the second section of this chapter we'll give an overview over the announced construction and over the proof of the main result.
Chapter 2 mostly consists of definitions, but also contains some useful results that we'll use later on. Chapter 3 provides the announced construction, while Chapter 4 is dedicated to the proof of our main result. In the last chapter, Chapter 5, we'll sum up our results and make suggestions for possible future work.

1. Motivation

Bounded domains in $\mathbb{C}^n$ are common objects of interest in complex analysis. Since balls in $\mathbb{C}^n$ are very well-behaved, it becomes a natural question to ask how "close" a given bounded domain is to being a ball. In [**1**], F. Deng, Q. Guan and L. Zhang introduced and studied the so called *squeezing function* s_D of a bounded domain D in $\mathbb{C}^n$, a continuous function on D taking values in $(0, 1]$, that in some sense measures how much D looks like a ball – the bigger $s_D(z)$ for some $z \in D$, the more D looks like a ball observed from z. The same authors proved the following theorem in [**2**]:

Theorem 1.1. *Let D be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-boundary. Then* $\lim_{z\to bD} s_D(z) = 1$.

Their proof of Theorem 1.1 relies heavily on (a special case of) a result by K. Diederich, J. E. Fornæss and E. F. Wold ([**3**, Theorem 1.1 on p. 1]). We state said special case:

Theorem 1.2. *Let Ω be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-boundary and let $p \in b\Omega$. Then there exists an open neighborhood $\widetilde{\Omega}$ of $\overline{\Omega}$ and a holomorphic embedding $f\colon \widetilde{\Omega} \to \mathbb{C}^n$ such that:*

- $f(p) = (0, \ldots, 0, 1)$,
- $f(\overline{\Omega}) \subseteq \overline{\mathbb{B}^n}$
- $f(\overline{\Omega}) \cap b\mathbb{B}^n = \{p\}$,

where $\mathbb{B}^n$ *denotes the open unit ball centered at* 0 *in* $\mathbb{C}^n$*.*

This means that every boundary point p of a bounded strongly pseudoconvex domain Ω with $\mathcal{C}^2$-boundary in $\mathbb{C}^n$ can be "exposed" by an injective holomorphic map defined on a neighborhood of $\overline{\Omega}$: All of $\overline{\Omega}$ is mapped to the interior of the ball $\mathbb{B}^n$, with the exception of the point p, which is mapped to the boundary of the ball.
Roughly speaking, this is achieved by stretching a small part of the domain, so that (the image of) p "touches" the boundary of a big ball containing Ω, while the rest of the domain remains (almost) unchanged. In order to find a holomorphic embedding like that, we construct closed sets $A, B, C \subseteq \overline{\Omega}$ as well as an injective holomorphic map γ defined on a neighborhood of A with the following properties:

- $p \in A$, $p \notin B$ and A is very small,
- $C = A \cap B \neq \emptyset$ and $A \cup B = \overline{\Omega}$,
- the set C separates $A \setminus B$ and $B \setminus A$ in the sense that $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$,
- γ is close to the identity on a neighborhood of C and stretches A, so that $\gamma(p)$ "touches" the boundary of a big ball containing Ω (in the sense that $\gamma(A)$ satisfies some convexity assumptions at $\gamma(p)$).

Such a map γ can be constructed using Mergelyan's Theorem. We are *not* done, since the map γ is only defined on a neighborhood of A and not on a neighborhood of the bigger set $\overline{\Omega}$.
Since γ is close to the identity on a neighborhood of C, one might try to use the solution to the first Cousin problem to obtain an additive splitting

$$\gamma - \mathrm{Id} = b - a$$

on a neighborhood of C, where a (resp. b) is defined on a neighborhood of A (resp. B) and close to 0 in supremum norm. This gives rise to a holomorphic map g, defined on a neighborhood of $\overline{\Omega}$, which equals $\gamma + a$ on a neighborhood of A and $\mathrm{Id} + b$ on a neighborhood of B. Since a and b are small, one might hope that g has all the desired properties – it leaves B almost unchanged and stretches a small part of $\overline{\Omega}$ containing p. Unfortunately g is not necessarily injective, so this approach won't work.
We can, however, construct a suitable holomorphic embedding f from γ with the help of a result by F. Forstnerič ([**4**, Theorem 8.7.2 on p. 359]); it basically says that – if γ is close enough to the identity on a neighborhood of C and (A, B) is

a Cartan pair – we can find a *compositional* splitting

$$\gamma = \beta \circ \alpha^{-1}$$

on a neighborhood of C, where α (resp. β) is defined, injective and holomorphic on a neighborhood of A (resp. B) and close to the identity there. Now we can construct a suitable map f by letting $f = \beta$ on a neighborhood of B and $f = \gamma \circ \alpha$ on a neighborhood of A. Since α and β are close to the identity on their respective domains, the map f will stretch the domain at p and leave B almost unchanged. f is injective when restricted to a neighborhood of A resp. B. To deduce global injectivity, we point out that f stretches the part of A which is close to p "away from the domain" and the set C separates $A \setminus B$ and $B \setminus A$, which can be used to deduce global injectivity, since α and β are close to the identity on their respective domains and γ is close to the identity on a neighborhood of C.

Under some additional assumptions on Ω, one can show that the map f from Theorem 1.2 can be chosen to be a global holomorphic automorphism of $\mathbb{C}^n$ ([**3**, Theorem 1.3 on p. 2]). By pulling back $T^{\mathbb{C}}_{f(p)}(b\mathbb{B}^n)$ with f one gets the existence of globally defined support surfaces which are closed smooth (complex) hypersurfaces, touching $b\Omega$ only from the outside at p (for details see [**3**]).

A question that arises in this context is, whether it is possible to choose a holomorphic embedding f_ζ as in Theorem 1.2 for each $\zeta \in b\Omega$, such that f_ζ depends continuously on ζ. It is also natural to ask about the existence of a smooth family of globally defined support surfaces as above (see [**3**, Theorem 1.4 on p. 2]).
Looking at the above rough sketch for the proof of Theorem 1.2, it is obvious that we need a result telling us that the maps obtained from our compositional splitting (α_ζ and β_ζ) can be chosen to depend continuously on the parameter (in this case the point $\zeta \in b\Omega$) – assuming that γ_ζ, A_ζ, B_ζ and C_ζ do. Finding and proving such a result is the goal of this thesis.

2. Overview

As mentioned in the previous section, the goal of this thesis is to find a parameter version of [**4**, Theorem 8.7.2 on p. 359] by F. Forstnerič. The main work is done in Chapter 3. There we develop a version of [**4**, Theorem 8.7.2 on p. 359] with arbitrary parameter spaces with a high degree of generality by (roughly) following F. Forstnerič's proof and ensuring continuous dependence on the parameter along the way. The theorem we end up with in Chapter 3 is Theorem 3.6.
In Chapter 4 we will apply Theorem 3.6 to the situation described in the previous section (roughly). What happens in Chapter 4 is more or less a technicality

(although the result we obtain there is the main result of this thesis), which is why we will concentrate on explaining the approach we take in Chapter 3 in this section:

Given γ_ζ defined (and close to the identity) on an open set containing $C_\zeta = A_\zeta \cap B_\zeta$, we use the solution operators to the $\overline{\partial}$-equation to find $a_{\zeta,0}$ and $b_{\zeta,0}$ (defined on open sets containig A_ζ resp. B_ζ), such that

$$\gamma_\zeta - \mathrm{Id} = b_{\zeta,0} - a_{\zeta,0}$$

on an open set containing C_ζ. This will be done in Lemma 3.14. The existence of the solution operators to the $\overline{\partial}$-equation will be one of the assumptions in Chapter 3. In Chapter 4 the existence of said operators will follow from some well-known results in several complex variables. We set

$$\alpha_{\zeta,0} := \mathrm{Id} + a_{\zeta,0},$$
$$\beta_{\zeta,0} := \mathrm{Id} + b_{\zeta,0},$$

and define

$$\gamma_{\zeta,1} := \beta_{\zeta,0}^{-1} \circ \gamma_\zeta \circ \alpha_{\zeta,0}$$

on an open set containing C_ζ. We iterate and obtain sequences $(\alpha_{\zeta,k})_{k\in\mathbb{Z}_{\geq 0}}$, $(\beta_{\zeta,k})_{k\in\mathbb{Z}_{\geq 0}}$ and $(\gamma_{\zeta,k})_{k\in\mathbb{Z}_{\geq 1}}$, such that

$$\gamma_{\zeta,k+1} = \beta_{\zeta,k}^{-1} \circ \gamma_{\zeta,k} \circ \alpha_{\zeta,k}$$

on an open set containing C_ζ. The estimates we will establish in Lemma 3.11 and in Lemma 3.14 allow us to show that everything is welldefined and that $(\gamma_{\zeta,k})_{k\in\mathbb{Z}_{\geq 1}}$ converges to the identity on some open set containing C_ζ, which in turn enables us to construct the desired compositional splitting of γ_ζ. Shrinking the occuring domains in a controlled way is essential both for convergence (and welldefinedness) of the above process and for the continuous dependence on the parameter ζ. This is made possible by the Cauchy estimates; we will give the details in Lemma 3.7.

3. Acknowledgment

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CHAPTER 2

Definitions and Preliminaries

In this chapter we will give a quick overview over the basic notions appearing in this thesis (most of them are taken from [**4**]) and state some known results involving those. It is assumed that the reader is familiar with the basic theory of several complex variables in $\mathbb{C}^n$. Sections 1, 2 and 3 will only be needed to state the original splitting lemma by F. Forstnerič ([**4**, Theorem 8.7.2 on p. 359]).

In order to avoid confusion we make the following remark concerning notation:

REMARK 2.1. From now on, in the spirit of [**8**], any open subset of $\mathbb{C}^n$ will be called a *domain*, i.e. domains are *not* assumed to be connected. It should be noted, however, that we *did* assume them to be connected in Chapter 1.

1. Complex Manifolds

This section mostly consists of definitions. We assume the reader to be familiar with the basic concepts from topology.

DEFINITION 2.2. A second countable Hausdorff space X is called a *topological manifold of dimension n* if every point $p \in X$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

DEFINITION 2.3. Let X be a topological manifold of dimension $2n$, where $n \in \mathbb{Z}_{\geq 0}$. A *complex atlas* on X is a collection $\mathcal{U} = \{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ having the following properties:

- $\forall \alpha \in A\colon U_\alpha$ is an open subset of X,
- $\bigcup_{\alpha \in A} U_\alpha = X$,
- $\forall \alpha \in A\colon \phi_\alpha$ is a homeomorphism from U_α onto an open subset of $\mathbb{R}^{2n} = \mathbb{C}^n$,
- $\forall \alpha, \beta \in A\colon$ The *transition map* $\phi_\alpha \circ {\phi_\beta}^{-1}\colon \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is biholomorphic.

An element (U_α, ϕ_α) of a complex atlas is called a *complex chart* or a *local holomorphic coordinate system* on X.

Definition 2.4. Let X be a topological manifold of dimension $2n$. Two complex atlases $\mathcal{U}$ and $\mathcal{V}$ on X are said to be *holomorphically compatible* if $\mathcal{U} \cup \mathcal{V}$ is also a complex atlas on X. This defines an equivalence relation on the set of complex atlases on X. An equivalence class with respect to this equivalence relation is called a *complex structure* on X. A complex structure $\mathcal{S}$ on X contains a uniquely determined complex atlas on X which is maximal with respect to inclusion; it is called the *maximal (complex) atlas* on X contained in $\mathcal{S}$ and equals the union of all the atlases contained in $\mathcal{S}$.

Definition 2.5. A *complex manifold of (complex) dimension* n is a topological manifold X of dimension $2n$ equipped with a complex structure on X.

Remark. Let X be a complex manifold. Whenever we consider a complex chart (U, ϕ) on X, we will (unless specified otherwise) assume (U, ϕ) to be contained in the maximal atlas contained in the complex structure X is equipped with.

Remark. If X is a complex manifold of dimension n then any nonempty open subset U of X is also a complex manifold of dimension n (with the canonical complex structure it inherits from X).

Remark. Unless stated otherwise, we will always assume complex manifolds to have *positive* (complex) dimension.

Definition 2.6. Let X be a complex manifold of dimension n. A function $f\colon X \to \mathbb{C}$ is said to be *holomorphic* if for all complex charts (U, ϕ) on X the function $f \circ \phi^{-1}\colon \phi(U) \to \mathbb{C}$ is holomorphic on the open subset $\phi(U)$ of $\mathbb{C}^n$. The set of all holomorphic functions on X will be denoted by $\mathcal{O}(X)$. Unless specified otherwise, $\mathcal{O}(X) \subseteq \mathcal{C}(X, \mathbb{C})$ will be equipped with the compact-open topology.

Definition 2.7. Let X be a complex manifold of dimension n and let Y be a complex manifold of dimension m. A continuous map $f\colon X \to Y$ is said to be a *holomorphic map* if for all points $p \in X$ there exists complex charts (U, ϕ) on X and (V, ψ) on Y, such that the following conditions are satisfied:

- $p \in U$,
- $f(U) \subseteq V$,
- $\psi \circ f \circ \phi^{-1}$ is a holomorphic map from the open subset $\phi(U)$ of $\mathbb{C}^n$ to the open subset $\psi(V)$ of $\mathbb{C}^m$.

Definition 2.8. Let X be a complex manifold of dimension n and let Y be a complex manifold of dimension m. A bijective map $f\colon X \to Y$ is called a *biholomorphism* if both f and its inverse $f^{-1}\colon Y \to X$ are holomorphic maps. In this case f is a homeomorphism and $n = m$.

Remark. A bijective holomorphic map $f\colon X \to Y$ from a complex manifold X to a complex manifold Y is a biholomorphism.

DEFINITION 2.9. Let X be a complex manifold of dimension n; let $M \subseteq X$ and $m \in \{0, 1, \ldots, n\}$. We say that M is a *complex submanifold of X of dimension m* if for every point $x \in X$ there exists a complex chart (U, ϕ) with $x \in U$, such that $\phi(U \cap M) = (\mathbb{C}^m \times \{0\}^{n-m}) \cap \phi(U)$. Any such chart is said to be *adapted* to M. If we denote the projection from $\mathbb{C}^n$ to the first m complex coordinates by π, then the collection

$$\{(U \cap M, \pi \circ \phi|_{U \cap M}) \colon (U, \phi) \text{ adapted to } M\}$$

is a complex atlas on M. The complex structure on M induced by this atlas is called the *complex submanifold structure induced by the inclusion $M \hookrightarrow X$.* Unless specified otherwise, we will always assume complex submanifolds to be equipped with the complex submanifold structure.

2. Stein Manifolds and Cartan Pairs

In this section we will introduce Stein manifolds and Cartan pairs. These concepts are indispensable for the proof of the original splitting lemma ([**4**, Theorem 8.7.2 on p. 359]) because of their connection to the $\overline{\partial}$-equation.

DEFINITION 2.10. Let X be a complex manifold and let K be a compact subset of X. Then the set

$$\widehat{K}_{\mathcal{O}(X)} := \{x \in X \colon |f(x)| \leq \sup_K |f| \text{ for every } f \in \mathcal{O}(X)\}$$

is called the *holomorphically convex hull* of K in X or the $\mathcal{O}(X)$*-hull* of K.

DEFINITION 2.11. Let X be a complex manifold and let K be a compact subset of X. The set K is called $\mathcal{O}(X)$*-convex* if $K = \widehat{K}_{\mathcal{O}(X)}$.

DEFINITION 2.12. A complex manifold X is called *holomorphically convex* if for every compact subset K of X the $\mathcal{O}(X)$-hull $\widehat{K}_{\mathcal{O}(X)}$ is also compact.

The following definition is taken from [**6**]:

DEFINITION 2.13. A complex manifold X is said to be *holomorphically spreadable* if for any point x_0 in X there are holomorphic functions $f_1, \ldots, f_N \colon X \to \mathbb{C}$, such that x_0 is (contained and) isolated in the set

$${f_1}^{-1}(\{0\}) \cap \ldots \cap {f_N}^{-1}(\{0\}).$$

DEFINITION 2.14. A complex manifold X of dimension n is said to be a *Stein manifold* or a *holomorphically complete manifold* if it has the following properties:

- X is holomorphically spreadable.
- X is holomorphically convex.

Remark. For readers familiar with the Zariski differential we provide an equivalent definition of Stein manifolds (the equivalence of those definitions was shown by Hans Grauert in [**7**]):

A complex manifold X of dimension n is said to be a *Stein manifold* or a *holomorphically complete manifold* if it has the following properties:

- If x and y are distinct points in X then there exists a holomorphic function $f\colon X \to \mathbb{C}$ satisfying $f(x) \neq f(y)$.
- If x is a point in X then there exist $f_1, \dots, f_n \in \mathcal{O}(X)$ whose differentials $\mathrm{d}f_1, \dots, \mathrm{d}f_n$ are $\mathbb{C}$-linearly independent at x.
- X is holomorphically convex.

Definition 2.15. Let X be a complex manifold and let K be a compact subset of X. Then K is called a *Stein compactum* if it admits a neighborhood basis of open subsets of X which are Stein manifolds.

Definition 2.16. Let X be a complex manifold and let A and B be compact subsets of X. The pair (A, B) is called a *Cartan pair* if it has the following properties:

- A, B, $A \cap B$ and $A \cup B$ are Stein compacta.
- A and B are separated in the sense that $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$.

3. Holomorphic Foliations

In this section we introduce (nonsingular) holomorphic foliations and a class of injective holomorphic maps preserving a given holomorphic foliation in some sense.

Definition 2.17. A subset S of a topological space X is called *locally closed* if it is the intersection of an open and a closed subset of X, i.e. if there exist an open subset U of X and a closed subset C of X satisfying $S = C \cap U$.

Definition 2.18. Let X be a complex manifold of dimension n and let $m \in \{1, \dots, n\}$. A *nonsingular holomorphic foliation* $\mathcal{F}$ of dimension m on X is a subdivision of X into m-dimensional, locally closed, connected complex submanifolds $F_\alpha \subseteq X, \alpha \in A$, called the *leafes* of $\mathcal{F}$, having the following properties:

- X is the disjoint union of the leafes, i.e. $X = \coprod_{\alpha \in A} F_\alpha$.
- For every point $p \in X$ there exist an open neighborhood $U \subseteq X$ and a biholomorphic map $f = (f_1, f_2) : U \to \mathbb{D}^m \times \mathbb{D}^{n-m} \subseteq \mathbb{C}^n$, such that for every $\alpha \in A$, the intersection $F_\alpha \cap U$ is a union of sets of the form $\{x \in U \colon f_2(x) = c\}$.

NOTE. As usual, $\mathbb{D}$ denotes the open unit disc in $\mathbb{C}$ centered at 0.

DEFINITION 2.19. Let X be a complex manifold of dimension n, let $\mathcal{F}$ be a nonsingular holomorphic foliation of dimension m on X and let $\{F_\alpha : \alpha \in A\}$ be the collection of leafes of $\mathcal{F}$. Let $\pi_2 : \mathbb{C}^n \to \mathbb{C}^{n-m}, (z_1, \dots, z_n) \mapsto (z_{m+1}, \dots, z_n)$ denote to projection to the last $(n-m)$ complex coordinates. A *distinguished chart* on X with respect to $\mathcal{F}$ is a complex chart (U, ϕ) on X with $\phi(U) = \mathbb{D}^n$, such that for every $\alpha \in A$, the intersection $F_\alpha \cap U$ is a union of sets of the form $\{x \in U : (\pi_2 \circ \phi)(x) = c\}$.

NOTE. In the situation of Definition 2.19, the elements of the collection

$$\{C \subseteq U : \text{there exists } \alpha \in A, \text{ s.t. } C \text{ is a connected component of } F_\alpha \cap U\}$$

will be called the *plaques* of U.

DEFINITION 2.20. Let X be a complex manifold of dimension n, let $\mathcal{F}$ be a nonsingular holomorphic foliation of dimension m on X and let $V \Subset X$. A finite collection $\mathcal{U} = \{(U_1, \phi_1), \dots, (U_N, \phi_N)\}$ of distinguished charts on X with respect to $\mathcal{F}$ is said to be a *regular $\mathcal{F}$-atlas on V* if there exists $0 < r < 1$, such that the following conditions are satisfied:

- $\overline{V} \subseteq \bigcup_{j=1}^N {\phi_j}^{-1}((r\mathbb{D}^m) \times \mathbb{D}^{n-m})$,
- If $j_1, j_2, j_3 \in \{1, \dots, N\}$, the set $\{j_1, j_2, j_3\}$ has cardinality 2 or 3 and $U_{j_1} \cup U_{j_2} \cup U_{j_3}$ is connected, then there exists a distinguished chart (U, ϕ), such that for all $k \in \{1, 2, 3\}$, each plaque of U meets at most one plaque of U_{j_k}.

DEFINITION 2.21. Let X be a complex manifold of dimension n, let $\mathcal{F}$ be a nonsingular holomorphic foliation of dimension m on X, let $V \Subset X$ be open and let $\gamma : V \to X$ be an injective holomorphic map. γ is said to be an *$\mathcal{F}$-map*, if there exists a regular $\mathcal{F}$-atlas $\mathcal{U} = \{(U_1, \phi_1), \dots, (U_N, \phi_N)\}$ on V (and an r as in Definition 2.20), such that for all $j \in \{1, \dots, N\}$ the restriction of γ to $V \cap {\phi_j}^{-1}((r\mathbb{D}^m) \times \mathbb{D}^{n-m})$ has image contained in U_j and is of the form $(z, w) \mapsto (c_j(z, w), w)$ in the distinguished holomorphic coordinates (z, w) on U_j.

4. The $\overline{\partial}$-Problem in $\mathbb{C}^n$

An important part in the proof of our splitting lemma will be establishing the existence of additive splittings of certain holomorphic maps. The crucial step in establishing an additive splitting will be solving the $\overline{\partial}$-equation. This section will be devoted to stating the results on solving the $\overline{\partial}$-equation that we are going to use in the proof of our main result. We'll limit ourselves to results in $\mathbb{C}^n$, since the more general results on complex manifolds won't be needed in this thesis.

The following theorem is a special case of [**8**, Theorem 2.7 on p. 203]:

THEOREM 2.22. *Let $D \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain with $\mathcal{C}^2$ boundary. There exists a linear operator*

$$S\colon \mathcal{C}_{0,1}(\overline{D}) \to \mathcal{C}^0(D)$$

and a constant $C > 0$ with the following properties:

- $\forall k \in \mathbb{Z}_{\geq 0}$: *If $f \in \mathcal{C}_{0,1}(\overline{D}) \cap \mathcal{C}^k_{0,1}(D)$, then $S(f) \in \mathcal{C}^k(D)$,*
- $|S(f)|_{1/2,D} \leq C|f|_{\overline{D}}$ *for all $f \in \mathcal{C}_{0,1}(\overline{D}) \cap \mathcal{C}^1_{0,1}(D)$,*
- *if $f \in \mathcal{C}^1_{0,1}(\overline{D})$ and $\overline{\partial} f = 0$, then $\overline{\partial}(S(f)) = f$.*

NOTE 2.23. $|S(f)|_{1/2,D}$ denotes the $\frac{1}{2}^{th}$-Hölder norm and *not* the $\frac{1}{2}^{th}$-Hölder seminorm in Theorem 2.22.

In the proof of our splitting lemma, the domains where we solve the $\overline{\partial}$-equation will vary. Hence the following result is important; it says that we can solve the $\overline{\partial}$-equation with the same constant $C > 0$ for all domains which are sufficiently small $\mathcal{C}^2$-perturbations of an initial domain. It is a special case of [**8**, Theorem 3.6 on p. 212]:

THEOREM 2.24. *Let $D \Subset \mathbb{C}^n$ be a strictly pseudoconvex domain with $\mathcal{C}^2$ boundary and with a strictly plurisubharmonic $\mathcal{C}^2$ defining function r_0, defined on a neighborhood U of bD, where r_0 satisfies $dr_0 \neq 0$. Then there are constants $C > 0$ and $\tau_0 > 0$, such that for any $r \in \{\rho \in \mathcal{C}^2(U, \mathbb{R})\colon |\rho - r_0|_{2,U} < \tau_0\}$ there is a linear operator $S^r\colon \mathcal{C}_{0,1}(\overline{D^r}) \to \mathcal{C}^0(D^r)$ as in Theorem 2.22, such that*

$$|S^r(f)|_{1/2,D^r} \leq C|f|_{\overline{D^r}}$$

for all $f \in \mathcal{C}_{0,1}(\overline{D^r}) \cap \mathcal{C}^1_{0,1}(D^r)$.

NOTE. In the situation of Theorem 2.24, the set D^r is of course defined as follows:

$$D^r = (D \setminus U) \cup \{x \in U\colon r(x) < 0\}.$$

5. The Signed Distance Function

In this section we introduce the signed distance function. It will be essential in the proof of our main result, since some of the domains occuring in that proof are level sets of the signed distance function.

DEFINITION 2.25. Let (X, d) be a metric space and let A be a nonempty proper subset of X. Then we define the *signed distance function* ρ_A as follows:

$$\rho_A\colon X \to \mathbb{R}$$
$$x \mapsto \begin{cases} -\inf_{b \in X\setminus A} d(x,b) & \text{if } x \in A \\ \inf_{a \in A} d(x,a) & \text{if } x \notin A \end{cases}$$

Note that ρ_A is welldefined because of the assumptions on A.

The following well-known property of the signed distance function is what makes it so useful for our purposes:

LEMMA 2.26. *Let Ω be a nonempty open subset of $\mathbb{R}^k$, where $k \geq 2$. Assume Ω is bounded and the boundary $b\Omega$ of Ω is of class $\mathcal{C}^2$. Then there is an open neighborhood U of $b\Omega$, where the signed distance function ρ_Ω (defined with respect to the euclidean metric) is $\mathcal{C}^2$-smooth and satisfies $\mathrm{d}\rho_\Omega(x) \neq 0$ for all $x \in U$.*

REMARK. In the situation of Lemma 2.26 the function ρ_Ω actually measures the distance of a point in $\mathbb{R}^k$ to the boundary $b\Omega$ of Ω and adds a sign, giving the boundary an orientation.

CHAPTER 3

Splitting Lemmata

In this chapter we will state and prove a splitting lemma on varying domains, which in turn will imply the main result of this thesis. We are interested in compositional splittings of biholomorphic maps close to the identity. The previously known results give the existence of such splittings on nonvarying domains, but for some applications a result on varying domains will be required. We derive such a result by adapting the proof of [**4**, Theorem 8.7.2 on p. 359] by F. Forstnerič to the situation of domains varying "pleasantly" with a parameter.

The following definition will make notation easier:

DEFINITION 3.1. Let X be a complex manifold of dimension n. Assume that dist is a distance function on X induced by a smooth Riemannian metric. If $\emptyset \neq V \subseteq X$ and $\gamma_1, \gamma_2 \colon V \to X$ are maps, then we set

$$\operatorname{dist}_V(\gamma_1, \gamma_2) := \sup_{x \in V} \operatorname{dist}(\gamma_1(x), \gamma_2(x)) \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

If $X = \mathbb{C}^n$ we'll always assume that dist is the euclidean metric, unless stated otherwise.

1. A Splitting Lemma on a Nonvarying Domain

This section is devoted to stating a result found and proven by F. Forstnerič, giving a compositional splitting of biholomorphic maps close to the identity on an open set containing the intersection of the entries of a Cartan pair. This result is taken from [**4**, Theorem 8.7.2 on p. 359].

THEOREM 3.2. *Let X be a complex manifold, let* dist *be a distance function on X induced by a smooth Riemannian metric, let (A, B) be a Cartan pair in X and let $\widetilde{C}$ be an open subset of X containing $C = A \cap B$. Then there exist open subsets A', B' and C' of X, with $A \subseteq A'$, $B \subseteq B'$ and $C' \subseteq A' \cap B' \subseteq \widetilde{C}$, and an $\epsilon_\eta > 0$ for every $\eta > 0$, such that the following property is satisfied:*

For every injective holomorphic map $\gamma \colon \widetilde{C} \to X$ with $\operatorname{dist}_{\widetilde{C}}(\gamma, \mathrm{Id}) < \epsilon_\eta$ there exist injective holomorphic maps $\alpha \colon A' \to X$ and $\beta \colon B' \to X$, satisfying the following:

- *α and β depend continuously on γ,*
- *$\gamma = \beta \circ \alpha^{-1}$ on C',*
- *$\operatorname{dist}_{A'}(\alpha, \mathrm{Id}) < \eta$,*
- *$\operatorname{dist}_{B'}(\beta, \mathrm{Id}) < \eta$.*

If $\mathcal{F}$ is a nonsingular holomorphic foliation on X and γ is an $\mathcal{F}$-map on $\widetilde{C}$, then α and β can be chosen to be $\mathcal{F}$-maps on A' resp. B'. If furthermore X_0 is a closed complex subvariety of X that doesn't meet C, then we can choose α and β to be tangent to the identity map to any finite order along X_0.

Remark. In the situation of Theorem 3.2 the numbers ϵ_η may depend on the foliation $\mathcal{F}$.

2. Domains Varying with a Parameter

In this section we want to introduce domains that vary "pleasantly" with a parameter. That notion of *varying pleasantly* has to be defined in a way that allows us to adjust the proof of [**4**, Theorem 8.7.2 on p. 359] to our situation, so that the resulting maps depend continuously on the parameter. Since all the applications only require results in $\mathbb{C}^n$, we'll restrict ourselves to considering domains in $\mathbb{C}^n$ with $n \in \mathbb{Z}_{\geq 1}$.

Definition 3.3. Let M be a nonempty subset of $\mathbb{C}^n$ and let $r > 0$. Then we define:

$$M(r) := \{z \in \mathbb{C}^n : \exists x \in M \text{ s.t. } |x - z| < r\}.$$

$M(r)$ obviously is an open subset of $\mathbb{C}^n$.

Remark. Let M be a nonempty subset of $\mathbb{C}^n$ and let $r, r_1, r_2 > 0$. Then we obviously have:

- $M \Subset \mathbb{C}^n$ if and only if $M(r) \Subset \mathbb{C}^n$,
- If $M \Subset \mathbb{C}^n$, then $M \Subset M(r)$,
- $M(r_1 + r_2) = \big(M(r_1)\big)(r_2)$.

The following definition is very technical and at first glance doesn't seem to catch any notion of *continuously varying domains*, but it will turn out that the properties included in this definition are precisely what we need to obtain maps that depend continuously on the parameter.

Definition 3.4. Let $\mathcal{P} \neq \emptyset$ be a topological space (called *parameter space*). A pair $(\{(A_\zeta, B_\zeta)\}_{\zeta \in \mathcal{P}}, \widetilde{\tau})$ is called *pleasant* if the following conditions are satisfied:

(1) $\widetilde{\tau} \in \mathbb{R}_{>0}$ and for all $\zeta \in \mathcal{P}$: A_ζ and B_ζ are subsets of $\mathbb{C}^n$,

(2) for all $\zeta \in \mathcal{P}$: $C_\zeta := A_\zeta \cap B_\zeta \neq \emptyset$,
(3) $\bigcup_{\zeta\in\mathcal{P}} D_\zeta \Subset \mathbb{C}^n$, where $D_\zeta := A_\zeta \cup B_\zeta$ for all $\zeta \in \mathcal{P}$,
(4) there exists $\tau' > 0$ such that:
  (a) $\tau' < \widetilde{\tau}$,
  (b) for all $\tau \in (0,\tau'), \zeta \in \mathcal{P}$: $A_\zeta(4\widetilde{\tau}+\tau) \cap B_\zeta(4\widetilde{\tau}+\tau) = C_\zeta(4\widetilde{\tau}+\tau)$,
  (c) there are a constant $C > 0$ and a collection $\{S^{\zeta,\tau}\}_{\zeta\in\mathcal{P},\tau\in(0,\tau')}$ of linear operators $S^{\zeta,\tau}\colon \mathcal{C}_{0,1}(\overline{D_\zeta(4\widetilde{\tau}+\tau)}) \to \mathcal{C}^0(D_\zeta(4\widetilde{\tau}+\tau))$, satisfying the following (compare this to Remark 3.5):
    (i) $\forall k \in \mathbb{Z}_{\geq 0}$: If $f \in \mathcal{C}_{0,1}(\overline{D_\zeta(4\widetilde{\tau}+\tau)}) \cap \mathcal{C}^k_{0,1}(D_\zeta(4\widetilde{\tau}+\tau))$, then $S^{\zeta,\tau}(f) \in \mathcal{C}^k(D_\zeta(4\widetilde{\tau}+\tau))$,
    (ii) if $f \in \mathcal{C}_{0,1}(\overline{D_\zeta(4\widetilde{\tau}+\tau)}) \cap \mathcal{C}^1_{0,1}(D_\zeta(4\widetilde{\tau}+\tau))$, then we have (compare this to Note 2.23): $|S^{\zeta,\tau}(f)|_{1/2,D_\zeta(4\widetilde{\tau}+\tau)} \leq C|f|_{\overline{D_\zeta(4\widetilde{\tau}+\tau)}}$,
    (iii) if $f \in \mathcal{C}^1_{0,1}(\overline{D_\zeta(4\widetilde{\tau}+\tau)})$ and $\overline{\partial} f = 0$ on $D_\zeta(4\widetilde{\tau}+\tau)$, then $\overline{\partial}(S^{\zeta,\tau}(f)) = f$,
    (iv) for all $\tau \in (0,\tau')$ we have the following:
    If $f_\zeta = \sum_{k=1}^n f_k^{(\zeta)} \mathrm{d}\bar{z}_k \in \mathcal{C}^1_{0,1}(\overline{D_\zeta(4\widetilde{\tau}+\tau)})$ for all $\zeta \in \mathcal{P}$ and if the function $f_k\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in \overline{D_\zeta(4\widetilde{\tau}+\tau)}\} \to \mathbb{C}, (z,\zeta) \mapsto f_k^{(\zeta)}(z)$ is continuous for all $k \in \{1,\dots,n\}$, then the function $\mathcal{G}\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in D_\zeta(4\widetilde{\tau}+\tau)\} \to \mathbb{C}$, $(z,\zeta) \mapsto (S^{\zeta,\tau} f_\zeta)(z)$ is continuous.
  (d) There exists $\chi\colon \mathbb{C}^n \times \mathcal{P} \times (0,\tau') \to [0,1]$ with the following properties:
    (i) $\forall \zeta,\tau$: The map $\chi(\cdot,\zeta,\tau)$ is smooth on $\mathbb{C}^n$, $\equiv 1$ in a neighborhood of $\overline{A_\zeta(4\widetilde{\tau}+\tau) \setminus B_\zeta(4\widetilde{\tau}+\tau)}$ and $\equiv 0$ in a neighborhood of $\overline{B_\zeta(4\widetilde{\tau}+\tau) \setminus A_\zeta(4\widetilde{\tau}+\tau)}$,
    (ii) $\forall \tau$: The map $\chi(\cdot,\cdot,\tau)$ is continuous on $\mathbb{C}^n \times \mathcal{P}$,
    (iii) there exists a constant $K' > 0$, s.t. for all ζ,τ we have: $|\overline{\partial}(\chi(\cdot,\zeta,\tau))|_{C_\zeta(4\widetilde{\tau}+\tau)} < K'$,
    (iv) $\forall \tau \in (0,\tau'), j,k \in \{1,\dots,n\}$:
    If the map $c\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in C_\zeta(4\widetilde{\tau}+\tau)\} \to \mathbb{C}^n$ is continuous and if $c(\cdot,\zeta)$ is holomorphic and bounded on $C_\zeta(4\widetilde{\tau}+\tau)$ for all ζ, then the map
    $\Phi_{j,k,\tau}(c)\colon \{(p,\zeta)\colon p \in \overline{D_\zeta(4\widetilde{\tau}+3\tau/4)}\} \to \mathbb{C}$,
    $$(p,\zeta) \mapsto \begin{cases} c(p,\zeta)_j \cdot \frac{\partial(\chi(\cdot,\zeta,\tau))}{\partial \bar{z}_k}(p) & \text{if } p \in C_\zeta(4\widetilde{\tau}+\tau) \\ 0 & \text{otherwise} \end{cases}$$

is continuous. Here $c(p,\zeta)_j$ denotes the j-th component of $c(p,\zeta) \in \mathbb{C}^n$.

(v) $\forall \tau$: If the map $c\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in C_\zeta(4\widetilde{\tau}+\tau)\} \to \mathbb{C}^n$ is continuous and if $c(\cdot,\zeta)$ is holomorphic and bounded on $C_\zeta(4\widetilde{\tau}+\tau)$ for all ζ, then the maps Φ_1 and Φ_2 are continuous, where:

$\Phi_1\colon \{(z,\zeta)\colon z \in B_\zeta(4\widetilde{\tau}+\tau/2)\} \to \mathbb{C}^n$,

$$(z,\zeta) \mapsto \begin{cases} \chi(z,\zeta,\tau)\cdot c(z,\zeta) & \text{if } z \in C_\zeta(4\widetilde{\tau}+\tau) \\ 0 & \text{otherwise} \end{cases}$$

$\Phi_2\colon \{(z,\zeta)\colon z \in A_\zeta(4\widetilde{\tau}+\tau/2)\} \to \mathbb{C}^n$,

$$(z,\zeta) \mapsto \begin{cases} (\chi(z,\zeta,\tau)-1)\cdot c(z,\zeta) & \text{if } z \in C_\zeta(4\widetilde{\tau}+\tau) \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.5. In 4c in Definition 3.4 we establish the existence of solution operators to the $\overline{\partial}$-equation. Considering the signed distance function ρ_{D_ζ} and the function $\exp(A\cdot\rho_{D_\zeta})-1$ for large A, one might be tempted to get rid of the assumption 4c alltogether and instead (in the spirit of Theorem 2.22, Theorem 2.24 and Lemma 2.26) require that D_ζ is the closure of a connected strictly pseudoconvex domain with $\mathcal{C}^2$-boundary for all ζ and that ρ_{D_ζ} depends continuously on ζ (together with some other technical assumptions). The problem here is *not* ensuring the existence of the constant C from 4c in Definition 3.4 (although that might require some additional assumptions on $\mathcal{P}$, e.g. compactness), but property 4(c)iv in Definition 3.4: Theorem 2.24 only allows us to solve the $\overline{\partial}$-equation with the same constant, but does *not* give us that the solution operators depend continuously on the domain in some sense. One might try to look at the proof of Theorem 2.22 and deduce such a continuous dependence. Assuming that this works out, one can replace 4c in Definition 3.4 by a much more intuitive assumption.

Notation. If $(\{(A_\zeta,B_\zeta)\}_{\zeta\in\mathcal{P}},\widetilde{\tau})$ is pleasant, then we will always adapt the notation of Definition 3.4, unless stated otherwise, i.e. C_ζ, D_ζ, τ', $S^{\zeta,\tau}$, C, χ and K' are as in Definition 3.4.

3. A Splitting Lemma on Varying Domains

With the definitions made in Section 2 we're finally able to formulate the announced result on varying domains.

Theorem 3.6. *If $(\{(A_\zeta,B_\zeta)\}_{\zeta\in\mathcal{P}},\widetilde{\tau})$ is pleasant, then for each $\eta\in\mathbb{R}_{>0}$ there exists $\epsilon_\eta\in\mathbb{R}_{>0}$ such that:*

If $\mu > 5\widetilde{\tau}$ and if $\{\gamma_\zeta\}_{\zeta\in\mathcal{P}}$ is a family of injective holomorphic maps $\gamma_\zeta\colon C_\zeta(\mu) \to \mathbb{C}^n$ satisfying

- $\gamma\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in C_\zeta(\mu)\} \to \mathbb{C}^n$, $(z,\zeta) \mapsto \gamma_\zeta(z)$ *is continuous,*
- $\mathrm{dist}_{C_\zeta(\mu)}(\gamma_\zeta, \mathrm{Id}) < \epsilon_\eta$ *for all* $\zeta \in \mathcal{P}$*,*

then there exist families $\{\alpha_\zeta\}_{\zeta\in\mathcal{P}}$ and $\{\beta_\zeta\}_{\zeta\in\mathcal{P}}$ of injective holomorphic maps $\alpha_\zeta\colon A_\zeta(2\widetilde{\tau}) \to \mathbb{C}^n$ and $\beta_\zeta\colon B_\zeta(2\widetilde{\tau}) \to \mathbb{C}^n$ having the following properties:

(1) For all $\zeta \in \mathcal{P}$ we have $\gamma_\zeta = \beta_\zeta \circ {\alpha_\zeta}^{-1}$ on $C_\zeta(\widetilde{\tau})$,
(2) $\mathrm{dist}_{A_\zeta(2\widetilde{\tau})}(\alpha_\zeta, \mathrm{Id}) < \eta$ and $\mathrm{dist}_{B_\zeta(2\widetilde{\tau})}(\beta_\zeta, \mathrm{Id}) < \eta$,
(3) The maps α and β are continuous, where

$$\alpha\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in A_\zeta(2\widetilde{\tau})\} \to \mathbb{C}^n, (z,\zeta) \mapsto \alpha_\zeta(z),$$
$$\beta\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in B_\zeta(2\widetilde{\tau})\} \to \mathbb{C}^n, (z,\zeta) \mapsto \beta_\zeta(z).$$

4. Proof of the Splitting Lemma on Varying Domains

This section is devoted to proving the splitting lemma on varying domains formulated in Section 3. We start off by formulating and proving a couple of useful lemmata.

LEMMA 3.7. *There exists a constant $K > 0$, depending only on $n \in \mathbb{Z}_{\geq 1}$, with the following property:*

If D is a nonempty open subset of $\mathbb{C}^n$, $\delta > 0$ and $c\colon D(\delta) \to \mathbb{C}^n$ is a holomorphic mapping with $||c||_{D(\delta)} \leq K \cdot \delta$, then the following map is (holomorphic and) injective:

$$\mathcal{C}\colon D \to \mathbb{C}^n, z \mapsto z + c(z).$$

PROOF. Let K be a positive real number with $K < \frac{1}{16n^{5/2}}$. We'll show that K has the desired property. To this end let D, c, δ and $\mathcal{C}$ be as above. We have to show that $\mathcal{C}$ is injective.

Assume for the sake of a contradiction that there are $a, b \in D$ with $a \neq b$ and $\mathcal{C}(a) = \mathcal{C}(b)$. This implies

$$||a-b|| = ||c(a) - c(b)|| \leq ||c(a)|| + ||c(b)|| \leq 2||c||_{D(\delta)} \leq 2K\delta < \frac{\delta}{2},$$

so the real line segment $\mathcal{L} = \{a + t\cdot(b-a)\colon t \in [0,1]\}$ between a and b is contained in $D(\frac{\delta}{2})$. Hence if $p \in \mathcal{L}$, the polydisc with multiradius $(\frac{\delta}{2\sqrt{n}}, \ldots, \frac{\delta}{2\sqrt{n}})$ centered at p is contained in $D(\delta)$. So, for $k, l \in \{1, \dots, n\}$, the Cauchy estimates yield:

$$\begin{aligned} \left| \frac{\partial c_k}{\partial z_l}(p) \right| &\leq \frac{2\sqrt{n}}{\delta} \cdot |c_k|_{D(\delta)} \\ &\leq \frac{2\sqrt{n}}{\delta} \cdot ||c||_{D(\delta)} \\ &\leq \frac{2\sqrt{n}}{\delta} \cdot K\delta \\ &= 2K\sqrt{n}. \end{aligned}$$

For $k = 1, \dots, n$ define $\phi_k \colon [0,1] \to \mathbb{C}, t \mapsto c_k(a + t \cdot (b-a))$. Denoting the two real component functions of ϕ_k by $\phi_{k,1}$ resp. $\phi_{k,2}$, we can apply the mean value theorem to obtain $\xi_{k,1}, \xi_{k,2} \in (0,1)$ with

$$c_k(b) - c_k(a) = \phi_k(1) - \phi_k(0) = \begin{pmatrix} \phi_{k,1}(1) - \phi_{k,1}(0) \\ \phi_{k,2}(1) - \phi_{k,2}(0) \end{pmatrix} = \begin{pmatrix} \phi'_{k,1}(\xi_{k,1}) \\ \phi'_{k,2}(\xi_{k,2}) \end{pmatrix}.$$

For $k \in \{1, \dots, n\}, j \in \{1, 2\}$ we have by the chain rule:

$$\phi'_{k,j}(\xi_{k,j}) = \begin{pmatrix} \frac{\partial c_{k,j}}{\partial x_1} & \frac{\partial c_{k,j}}{\partial y_1} & \cdots & \frac{\partial c_{k,j}}{\partial x_n} & \frac{\partial c_{k,j}}{\partial y_n} \end{pmatrix} (a + \xi_{k,j} \cdot (b-a)) \cdot \begin{pmatrix} \mathrm{Re}(b_1 - a_1) \\ \mathrm{Im}(b_1 - a_1) \\ \vdots \\ \mathrm{Re}(b_n - a_n) \\ \mathrm{Im}(b_n - a_n) \end{pmatrix}.$$

Setting $p_{k,j} := a + \xi_{k,j} \cdot (b-a) \in \mathcal{L}$ and using the Cauchy-Schwarz inequality and the Cauchy-Riemann equations we deduce

$$\begin{aligned} |\phi'_{k,j}(\xi_{k,j})| &\leq \left\| \begin{pmatrix} \frac{\partial c_{k,j}}{\partial x_1} \\ \frac{\partial c_{k,j}}{\partial y_1} \\ \vdots \\ \frac{\partial c_{k,j}}{\partial x_n} \\ \frac{\partial c_{k,j}}{\partial y_n} \end{pmatrix} (p_{k,j}) \right\| \cdot \|b-a\| \\ &\leq \left\| \begin{pmatrix} \frac{\partial c_k}{\partial x_1} \\ \frac{\partial c_k}{\partial y_1} \\ \vdots \\ \frac{\partial c_k}{\partial x_n} \\ \frac{\partial c_k}{\partial y_n} \end{pmatrix} (p_{k,j}) \right\| \cdot \|b-a\| \\ &\leq \left(\sum_{l=1}^{n} \left\| \frac{\partial c_k}{\partial x_l}(p_{k,j}) \right\| + \left\| \frac{\partial c_k}{\partial y_l}(p_{k,j}) \right\| \right) \cdot \|b-a\| \end{aligned}$$

$$
\begin{aligned}
&= 2 \cdot \left(\sum_{l=1}^{n} \left\| \frac{\partial c_k}{\partial z_l}(p_{k,j}) \right\| \right) \cdot \|b-a\| \\
&\leq 2 \cdot \left(\sum_{l=1}^{n} 2K\sqrt{n} \right) \cdot \|b-a\| \\
&= 4Kn^{3/2} \cdot \|b-a\| .
\end{aligned}
$$

This in turn implies the following:

$$
\begin{aligned}
\|b-a\| &= \|c(b)-c(a)\| \\
&= \left\| \begin{pmatrix} \phi_{1,1}(1)-\phi_{1,1}(0) \\ \phi_{1,2}(1)-\phi_{1,2}(0) \\ \vdots \\ \phi_{n,1}(1)-\phi_{n,1}(0) \\ \phi_{n,2}(1)-\phi_{n,2}(0) \end{pmatrix} \right\| \\
&= \left\| \begin{pmatrix} \phi'_{1,1}(\xi_{1,1}) \\ \phi'_{1,2}(\xi_{1,2}) \\ \vdots \\ \phi'_{n,1}(\xi_{n,1}) \\ \phi'_{n,2}(\xi_{n,2}) \end{pmatrix} \right\| \\
&\leq \sum_{k=1}^{n} \sum_{j=1}^{2} |\phi'_{k,j}(\xi_{k,j})| \\
&\leq \sum_{k=1}^{n} \sum_{j=1}^{2} 4Kn^{3/2} \cdot \|b-a\| \\
&= 8Kn^{5/2} \cdot \|b-a\| \\
&\leq \frac{1}{2} \cdot \|b-a\|.
\end{aligned}
$$

Since $a \neq b$, we can divide by $\|b-a\|$ and obtain $1 \leq \frac{1}{2}$, which is a contradiction. $\square$

Remark 3.8. The map $c + \mathrm{Id}$ is injective on D if c is "small enough" (in euclidean norm) on the bigger domain $D(\delta)$. That alone is not sufficient for our purposes, since we will find ourselves in a situation where we have to shrink some domains "in a controlled way". This "controlled shrinking" will be possible because Lemma 3.7 gives us that the estimate $||c||_{D(\delta)} \leq K \cdot \delta$ is sufficient to ensure injectivity of $c + \mathrm{Id}$ on D, for some K that doesn't depend on D and δ.

Remark. One might hope to obtain a result implying that $c+\mathrm{Id}$ is injective on D, if $c\colon D \to \mathbb{C}^n$ is holomorphic and "small enough" in euclidean norm on D

(where D is a nonempty open subset of $\mathbb{C}^n$ with $D \Subset \mathbb{C}^n$). In fact, even in the case $D = \mathbb{D}$ (where $\mathbb{D}$ is the open unit disc centered at 0 in $\mathbb{C}$) and $n = 1$ this is not possible:

Claim. *For all $\epsilon > 0$ there exists a holomorphic map $c\colon \mathbb{D} \to \mathbb{C}$ with $||c||_{\mathbb{D}} < \epsilon$, such that $c + \mathrm{Id}$ is not injective on $\mathbb{D}$.*

Proof. For all $\epsilon > 0$ consider the map $c_\epsilon\colon \mathbb{D} \to \mathbb{C}$, given by:

$$z \mapsto -\epsilon \cdot \sum_{j=3}^{\infty} \frac{1}{j^2} z^j.$$

c_ϵ is (welldefined and) holomorphic and satisfies

$$||c_\epsilon||_{\mathbb{D}} \leq \epsilon \cdot \sum_{j=3}^{\infty} \frac{1}{j^2} < \epsilon$$

for all $\epsilon > 0$. Define $f_\epsilon\colon \mathbb{D} \to \mathbb{C}, z \mapsto z + c_\epsilon(z)$. We have to prove that f_ϵ is not injective. To this end define $h_\epsilon\colon (0,1) \to \mathbb{R}, x \mapsto f_\epsilon(x)$; this is obviously welldefined. It suffices to prove that h_ϵ is not injective. h_ϵ is smooth, since f_ϵ is holomorphic. For all $x \in (0,1)$ we have:

$$h_\epsilon'(x) = 1 + c_\epsilon'(x) = 1 - \epsilon \cdot \sum_{j=2}^{\infty} \frac{1}{j+1} x^j.$$

c_ϵ' is continuous and $c_\epsilon'(0) = 0$, so there exists $a \in (0,1)$ with $h_\epsilon'(a) > 0$. Furthermore we have

$$\lim_{x \in (0,1), x \to 1} c_\epsilon'(x) = -\infty,$$

so there exists $b \in (0,1)$ with $a < b$ and $h_\epsilon'(b) < 0$. We know that h_ϵ' is continuous and strictly decreasing, so there exists a *uniquely determined* $\xi \in (0,1)$ with $a < \xi < b$ and $h_\epsilon'(\xi) = 0$ by the intermediate value theorem. $h_\epsilon'' < 0$ on $(0,1)$, so h_ϵ has a local maximum at ξ. We have $h_\epsilon' > 0$ on $(0,\xi)$ and $h_\epsilon' < 0$ on $(\xi, 1)$, so h_ϵ has a *unique* global maximum (which it attains at ξ). Now pick some $v \in \mathbb{R}$ with

$$h_\epsilon(a), h_\epsilon(b) < v < h_\epsilon(\xi)$$

and use the intermediate value theorem to find $s_1 \in (a, \xi)$ and $s_2 \in (\xi, b)$ with $h_\epsilon(s_1) = v$ and $h_\epsilon(s_2) = v$. Hence h_ϵ is not injective, since $s_1 < s_2$ and $h_\epsilon(s_1) = h_\epsilon(s_2)$. $\square$

Looking at the proof of Lemma 3.7 we immediately get the following result. It gives us a Lipschitz estimate for holomorphic mappings for points that can be connected by a real line segment without intersecting the boundary of the domain.

LEMMA 3.9. *Let V be a nonempty open subset of $\mathbb{C}^n$, let $d > 0$, let $x, y \in V$ and let $F\colon V \to \mathbb{C}^n$ be holomorphic and bounded. Assume that the real line segment $\mathcal{L} := \{tx + (1-t)y \colon t \in [0,1]\}$ between x and y satisfies $\mathcal{L}(d) \subseteq V$.*

Then we have:

$$\|F(y) - F(x)\| \leq 4n^{5/2} \cdot \frac{\|F\|_V}{d} \cdot \|y - x\|.$$

PROOF. Analogous to the proof of Lemma 3.7. □

We now prove an elementary result that will help us with the estimates in the proof of Theorem 3.6:

LEMMA 3.10. *There exists a map $\rho\colon \mathbb{R}_{>0} \times \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$ with the following property:*

If $(a, B, C) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 1}$ and if $(\epsilon_m)_{m \in \mathbb{Z}_{\geq 0}}$ is a sequence of non-negative real numbers satisfying

- $0 \leq \epsilon_0 < \rho(a, B, C)$,
- $\epsilon_{m+1} \leq C \cdot \frac{2^{3m} {\epsilon_m}^2}{a}$ *for all* $m \in \mathbb{Z}_{\geq 0}$,

then we have for all $m \in \mathbb{Z}_{\geq 0}$:

$$16B\epsilon_m < \frac{a}{2^{3m}}.$$

The proof is just a basic calculation; we'll do it here for the sake of completeness:

PROOF. Define $\rho\colon \mathbb{R}_{>0} \times \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 1} \to \mathbb{R}_{>0}$ by:

$$(a, B, C) \mapsto \min\left\{\frac{a}{8C}, \frac{a}{16B}\right\}.$$

We have to show that ρ has the desired property. To this end let $(a, B, C) \in \mathbb{R}_{>0} \times \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 1}$ and let $(\epsilon_m)_{m \in \mathbb{Z}_{\geq 0}}$ be a sequence of non-negative real numbers satisfying

- $0 \leq \epsilon_0 < \rho(a, B, C)$,
- $\epsilon_{m+1} \leq C \cdot \frac{2^{3m} {\epsilon_m}^2}{a}$ for all $m \in \mathbb{Z}_{\geq 0}$.

Since $\rho(a,B,C) \leq a/(16B)$, it suffices to prove that $\epsilon_m < \rho(a,B,C)/8^m$ for all $m \in \mathbb{Z}_{\geq 0}$. For $m = 0$ this is true by assumption. We'll proceed inductively, i.e. assume $m \in \mathbb{Z}_{\geq 0}$ satisfies $\epsilon_m < \rho(a,B,C)/8^m$. We calculate:

$$\begin{aligned}\epsilon_{m+1} &\leq C \cdot \frac{2^{3m}{\epsilon_m}^2}{a} \\ &< \frac{8^m C}{a} \cdot \frac{\rho(a,B,C) \cdot \rho(a,B,C)}{8^m \cdot 8^m} \\ &\leq \frac{8^m C}{a} \cdot \frac{\rho(a,B,C) \cdot \frac{a}{8C}}{8^m \cdot 8^m} \\ &= \frac{\rho(a,B,C)}{8^{m+1}},\end{aligned}$$

as desired. This concludes the induction and hence the proof. □

The following lemma is a special case of [**4**, Lemma 8.7.4 on p. 360 and Remark 8.7.5 ond p. 362]. We will adapt the proof given in [**4**] to our situation and add some details to the original proof.

LEMMA 3.11. *There is a constant $M_2 \geq 1$, depending only on $n \in \mathbb{Z}_{\geq 1}$, such that the following holds:*

If V is a nonempty open subset of $\mathbb{C}^n$ and if $\epsilon, \delta \in \mathbb{R}$ satisfy $0 < \epsilon < \frac{\delta}{4}$ and $\alpha, \beta, \gamma \colon V(\delta) \to \mathbb{C}^n$ are injective holomorphic mappings which are ϵ-close to the identity on $V(\delta)$ (i.e. $\operatorname{dist}_{V(\delta)}(\alpha, \mathrm{Id}) < \epsilon$, $\operatorname{dist}_{V(\delta)}(\beta, \mathrm{Id}) < \epsilon$ and $\operatorname{dist}_{V(\delta)}(\gamma, \mathrm{Id}) < \epsilon$), then the mapping

$$\widetilde{\gamma} := \beta^{-1} \circ \gamma \circ \alpha \colon V \to \mathbb{C}^n$$

is welldefined, injective and holomorphic. Writing

$$\begin{aligned}\alpha &= a + \mathrm{Id}_{V(\delta)}, & \gamma &= c + \mathrm{Id}_{V(\delta)}, \\ \beta &= b + \mathrm{Id}_{V(\delta)}, & \widetilde{\gamma} &= \widetilde{c} + \mathrm{Id}_V,\end{aligned}$$

we have

$$\|\widetilde{c} - (c + a - b)\|_V \leq M_2 \frac{\epsilon^2}{\delta}.$$

If furthermore $c = b - a$ on V, then we have

$$\|\widetilde{c}\|_V \leq M_2 \frac{\epsilon^2}{\delta}.$$

PROOF. An injective holomorphic map from an open subset of $\mathbb{C}^n$ to $\mathbb{C}^n$ is a biholomorphism onto its image, so if $\widetilde{\gamma}$ is welldefined, it will also be (injective and) holomorphic. α is ϵ-close to the identity, so $\gamma \circ \alpha$ is welldefined on V. So, in order to prove welldefinedness of $\widetilde{\gamma}$, it suffices to show that the image $\beta(V(\delta))$

of β contains $(\gamma \circ \alpha)(V)$. Since α and γ are ϵ-close to the identity, it suffices to show that $\beta(V(\delta))$ contains $V(2\epsilon)$.

To this end let $x \in V(2\epsilon)$. Define $\Omega := B^{(n)}(x, \epsilon)$, the open ball of radius ϵ around x in $\mathbb{C}^n$ with respect to the euclidean metric, and set

$$f\colon \overline{\Omega} \to \mathbb{C}^n, z \mapsto \beta(z).$$

Ω is an open, connected, bounded and nonempty subset of $\mathbb{C}^n$ and f is smooth. Consider

$$F\colon \overline{\Omega} \times [0,1] \to \mathbb{C}^n, (z,t) \mapsto tz + (1-t)f(z).$$

F is a smooth homotopy between f and $\mathrm{Id}_{\overline{\Omega}}$ and the mapping

$$\mathcal{H}\colon [0,1] \to \mathcal{C}^1(\overline{\Omega}; \mathbb{C}^n), t \mapsto F(\cdot, t)$$

is continuous, where $\mathcal{C}^1(\overline{\Omega}; \mathbb{C}^n)$ is equipped with the usual topology. β is an injective holomorphic mapping from an open subset of $\mathbb{C}^n$ to $\mathbb{C}^n$, so x is a regular value of both f and $\mathrm{Id}_{\overline{\Omega}}$. We have $x \notin F(\mathrm{b}\Omega, t)$ for all $t \in [0,1]$, since $\mathrm{dist}_{V(\delta)}(\beta, \mathrm{Id}) < \epsilon$. Hence the $\mathcal{C}^1$-mapping degrees of f and $\mathrm{Id}_{\overline{\Omega}}$ are (welldefined and) equal, i.e.

$$\deg(f, \Omega, x) = \deg(\mathrm{Id}_{\overline{\Omega}}, \Omega, x) = 1.$$

We conclude

$$x \in f(\Omega) \subseteq \beta(V(\delta)),$$

as desired. Hence $\widetilde{\gamma}$ is welldefined, injective and holomorphic.

It remains to show the two estimates. The second one it obvious from the first one, so we only have to show the first estimate. Using Lemma 3.9 and the fact that the image of β contains $V(2\epsilon)$, we calculate (noting that all the occuring compositions are indeed welldefined on the respective sets):

$$\begin{aligned}
\|\widetilde{c} - (c + a - b)\|_V &= \|(\mathrm{Id} + \widetilde{c}) - (\mathrm{Id} + c + a - b)\|_V \\
&= \left\|(\beta^{-1} \circ \gamma \circ \alpha) - (\mathrm{Id} + c + a - b)\right\|_V \\
&\leq \left\|(\beta^{-1} \circ \gamma \circ \alpha) - ((\mathrm{Id} - b) \circ \gamma \circ \alpha)\right\|_V \\
&\quad + \|((\mathrm{Id} - b) \circ \gamma \circ \alpha) - (\mathrm{Id} + c + a - b)\|_V \\
&\leq \left\|\beta^{-1} - (\mathrm{Id} - b)\right\|_{V(2\epsilon)} \\
&\quad + \|((\mathrm{Id} - b) \circ \gamma \circ \alpha) - (\mathrm{Id} + c + a - b)\|_V \\
&= \|\mathrm{Id} - ((\mathrm{Id} - b) \circ \beta)\|_{\beta^{-1}(V(2\epsilon))} \\
&\quad + \|(\gamma \circ \alpha) - (b \circ \gamma \circ \alpha) - \mathrm{Id} - c - a + b\|_V \\
&= \|\mathrm{Id} - \beta + (b \circ \beta)\|_{\beta^{-1}(V(2\epsilon))} \\
&\quad + \|\mathrm{Id} + a + (c \circ \alpha) - (b \circ \gamma \circ \alpha) - \mathrm{Id} - c - a + b\|_V \\
&= \|-b + (b \circ \beta)\|_{\beta^{-1}(V(2\epsilon))}
\end{aligned}$$

$$
\begin{aligned}
&+ \|(c \circ \alpha) - (b \circ \gamma \circ \alpha) - c + b\|_V \\
\leq\ & \|(b \circ \beta) - b\|_{\beta^{-1}(V(2\epsilon))} \\
&+ \|(c \circ \alpha) - c\|_V \\
&+ \|(b \circ \gamma \circ \alpha) - b\|_V \\
\leq\ & 4n^{5/2} \cdot \frac{\|b\|_{V(\delta)}}{\delta - 3\epsilon} \cdot \mathrm{dist}_{\beta^{-1}(V(2\epsilon))}(\beta, \mathrm{Id}) \\
&+ 4n^{5/2} \cdot \frac{\|c\|_{V(\delta)}}{\delta - \epsilon} \cdot \mathrm{dist}_V(\alpha, \mathrm{Id}) \\
&+ 4n^{5/2} \cdot \frac{\|b\|_{V(\delta)}}{\delta - 2\epsilon} \cdot \mathrm{dist}_V(\gamma \circ \alpha, \mathrm{Id}) \\
\leq\ & 4n^{5/2} \cdot \left(\frac{\epsilon}{\delta - 3\epsilon} \cdot \epsilon + \frac{\epsilon}{\delta - \epsilon} \cdot \epsilon + \frac{\epsilon}{\delta - 2\epsilon} \cdot 2\epsilon \right) \\
\leq\ & 4n^{5/2} \cdot \left(\frac{2\epsilon^2}{\delta - 3\epsilon} + \frac{2\epsilon^2}{\delta - 3\epsilon} + \frac{2\epsilon^2}{\delta - 3\epsilon} \right) \\
=\ & 24n^{5/2} \cdot \frac{\epsilon^2}{\delta - 3\epsilon} \\
\leq\ & 24n^{5/2} \cdot \frac{\epsilon^2}{\delta - \frac{3}{4}\delta} \\
=\ & 96n^{5/2} \cdot \frac{\epsilon^2}{\delta},
\end{aligned}
$$

and we are done. □

Looking at the first part of the proof of Lemma 3.11, we immediately deduce the following lemma:

LEMMA 3.12. *Let D be a nonempty open subset of $\mathbb{C}^n$ and let $\epsilon, \delta \in \mathbb{R}$ satisfy $0 < \epsilon < \delta$. Assume $\Phi\colon D(\delta) \to \mathbb{C}^n$ is an injective holomorphic mapping which is ϵ-close to the identity on $D(\delta)$. Then we have:*

$$D(\delta - \epsilon) \subseteq \Phi(D(\delta)).$$

PROOF. Analogous to the proof of Lemma 3.11. □

NOTE. Of course we could significantly weaken the assumptions in Lemma 3.12, but since we won't need a more general result, we will leave them as they are.

Now we will use the solution operators to the $\overline{\partial}$-equation to establish the announced additive splitting. To make notation easier we'll start off with a definition:

DEFINITION 3.13. Let U be a nonempty open subset of $\mathbb{C}^n$. Then we denote the collection of bounded holomorphic mappings from U to $\mathbb{C}^n$ as follows:

$$\mathrm{HB}(U) := \{\Phi\colon U \to \mathbb{C}^n\colon \Phi \text{ is holomorphic and bounded}\}.$$

The following lemma is based on [**4**, Lemma 8.7.6 on p. 362]. We follow the idea of the proof given there and adapt it to our situation.

LEMMA 3.14. *If $(\{(A_\zeta, B_\zeta)\}_{\zeta\in\mathcal{P}}, \widetilde{\tau})$ is pleasant then there exist a constant $M_3 \geq 1$ and operators*

$$\begin{aligned}\mathcal{E}_\zeta^\tau &\colon \mathrm{HB}(C_\zeta(4\widetilde{\tau}+\tau)) \to \mathrm{HB}(A_\zeta(4\widetilde{\tau}+\tau/2)),\\ \mathcal{Z}_\zeta^\tau &\colon \mathrm{HB}(C_\zeta(4\widetilde{\tau}+\tau)) \to \mathrm{HB}(B_\zeta(4\widetilde{\tau}+\tau/2)),\end{aligned}$$

where $\zeta \in \mathcal{P}, \tau \in (0,\tau')$ and τ' is as in Definition 3.4, such that the following properties are fulfilled:

(1) If $c \in \mathrm{HB}(C_\zeta(4\widetilde{\tau}+\tau))$, where $\zeta \in \mathcal{P}$ and $\tau \in (0,\tau')$, then we have on $C_\zeta(4\widetilde{\tau}+\tau/2)$:

$$c = \mathcal{Z}_\zeta^\tau(c) - \mathcal{E}_\zeta^\tau(c)$$

(2) If $\zeta \in \mathcal{P}$ and $\tau \in (0,\tau')$ are fixed, then $\mathcal{E}_\zeta^\tau$ and $\mathcal{Z}_\zeta^\tau$ are linear operators satisfying:

$$\begin{aligned}\left\|\mathcal{E}_\zeta^\tau(c)\right\|_{A_\zeta(4\widetilde{\tau}+\tau/2)} &\leq M_3 \cdot \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)},\\ \left\|\mathcal{Z}_\zeta^\tau(c)\right\|_{B_\zeta(4\widetilde{\tau}+\tau/2)} &\leq M_3 \cdot \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)}\end{aligned}$$

for all $c \in \mathrm{HB}(C_\zeta(4\widetilde{\tau}+\tau))$.

(3) Let $\tau \in (0,\tau')$ be fixed.
If $c\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in C_\zeta(4\widetilde{\tau}+\tau)\} \to \mathbb{C}^n$ is continuous and $c(\cdot,\zeta) \in \mathrm{HB}(C_\zeta(4\widetilde{\tau}+\tau))$ for all $\zeta \in \mathcal{P}$, then the following two maps are (welldefined and) continuous:

$$\begin{aligned}a\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in A_\zeta(4\widetilde{\tau}+\tau/2)\} &\to \mathbb{C}^n,\\ (z,\zeta) &\mapsto \big(\mathcal{E}_\zeta^\tau(c(\cdot,\zeta))\big)(z),\\ b\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in B_\zeta(4\widetilde{\tau}+\tau/2)\} &\to \mathbb{C}^n,\\ (z,\zeta) &\mapsto \big(\mathcal{Z}_\zeta^\tau(c(\cdot,\zeta))\big)(z).\end{aligned}$$

PROOF. We'll make use of 4b in Definition 3.4 several times over the course of this proof without mentioning it every time. We adopt the notation from Definition 3.4. Fix $\zeta \in \mathcal{P}$ and $\tau \in (0,\tau')$. We'll construct the operators $\mathcal{E}_\zeta^\tau$ and $\mathcal{Z}_\zeta^\tau$. To this end let $c \in \mathrm{HB}(C_\zeta(4\widetilde{\tau}+\tau))$.

For each $j \in \{1, \dots, n\}$ we define a $(0,1)$-form f_j on $D_\zeta(4\widetilde{\tau} + \tau)$ as follows:

$$f_j := \begin{cases} \overline{\partial}\Big(\chi(\cdot, \zeta, \tau) \cdot c_j\Big) & \text{on } C_\zeta(4\widetilde{\tau} + \tau), \\ 0 = \sum_{k=1}^{n} 0 \mathrm{d}\bar{z}_k & \text{on } D_\zeta(4\widetilde{\tau} + \tau) \setminus C_\zeta(4\widetilde{\tau} + \tau). \end{cases}$$

Here c_j denotes the j-th (complex) component function of c. The form f_j is welldefined by 4(d)i in Definition 3.4. We want to show that f_j is smooth on $D_\zeta(4\widetilde{\tau} + \tau)$. To this end let $x \in D_\zeta(4\widetilde{\tau} + \tau)$. It suffices to prove that f_j is smooth in a neighborhood of x.

If $x \in C_\zeta(4\widetilde{\tau} + \tau)$, then this is clear. If $x \in \overline{B_\zeta(4\widetilde{\tau} + \tau) \setminus A_\zeta(4\widetilde{\tau} + \tau)}$, then $\chi(\cdot, \zeta, \tau)$ is $\equiv 0$ in a neighborhood U of x (by 4(d)i in Definition 3.4), so f_j is $\equiv 0$ (and hence smooth) in $U \cap D_\zeta(4\widetilde{\tau} + \tau)$. If $x \notin C_\zeta(4\widetilde{\tau} + \tau)$ and $x \notin \overline{B_\zeta(4\widetilde{\tau} + \tau) \setminus A_\zeta(4\widetilde{\tau} + \tau)}$, then we have $x \in \overline{A_\zeta(4\widetilde{\tau} + \tau) \setminus B_\zeta(4\widetilde{\tau} + \tau)}$. In this case $\big(\chi(\cdot, \zeta, \tau) - 1\big)$ is $\equiv 0$ in a neighborhood V of x (by 4(d)i in Definition 3.4). On $C_\zeta(4\widetilde{\tau} + \tau)$ we have

$$\overline{\partial}\Big(\chi(\cdot, \zeta, \tau) \cdot c_j\Big) = \overline{\partial}\Big(\big(\chi(\cdot, \zeta, \tau) - 1\big) \cdot c_j\Big),$$

since c_j is holomorphic on $C_\zeta(4\widetilde{\tau} + \tau)$; so f_j is $\equiv 0$ (and hence smooth) in $V \cap D_\zeta(4\widetilde{\tau} + \tau)$. We conclude that f_j is smooth on $D_\zeta(4\widetilde{\tau} + \tau)$.

For all j we get (by restricting f_j):

$$f_j \in \mathcal{C}^\infty_{0,1}\left(\overline{D_\zeta\left(4\widetilde{\tau} + \frac{3}{4}\tau\right)}\right).$$

We have $\overline{\partial} f_j = 0$, which can be seen analogously to the smoothness of f_j. Set $g_j := S^{\zeta, \frac{3}{4}\tau}(f_j)$. By 4(c)i and 4(c)iii in Definition 3.4 we have $g_j \in \mathcal{C}^\infty\big(D_\zeta\left(4\widetilde{\tau} + \frac{3}{4}\tau\right)\big)$ and $\overline{\partial} g_j = f_j$ on $D_\zeta\left(4\widetilde{\tau} + \frac{3}{4}\tau\right)$. We set

$$g := \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}.$$

Now we define $\mathcal{Z}^\tau_\zeta(c) \colon B_\zeta(4\widetilde{\tau} + \tau/2) \to \mathbb{C}^n$ as follows:

$$\mathcal{Z}^\tau_\zeta(c) := \begin{cases} \chi(\cdot, \zeta, \tau) \cdot c - g & \text{on } C_\zeta(4\widetilde{\tau} + \tau) \cap B_\zeta(4\widetilde{\tau} + \tau/2), \\ -g & \text{on } B_\zeta(4\widetilde{\tau} + \tau/2) \setminus C_\zeta(4\widetilde{\tau} + \tau). \end{cases}$$

Analogously we define $\mathcal{E}^\tau_\zeta(c) \colon A_\zeta(4\widetilde{\tau} + \tau/2) \to \mathbb{C}^n$ as follows:

$$\mathcal{E}^\tau_\zeta(c) := \begin{cases} \Big(\chi(\cdot, \zeta, \tau) - 1\Big) \cdot c - g & \text{on } C_\zeta(4\widetilde{\tau} + \tau) \cap A_\zeta(4\widetilde{\tau} + \tau/2), \\ -g & \text{on } A_\zeta(4\widetilde{\tau} + \tau/2) \setminus C_\zeta(4\widetilde{\tau} + \tau). \end{cases}$$

$\mathcal{Z}^\tau_\zeta(c)$ and $\mathcal{E}^\tau_\zeta(c)$ are welldefined and smooth on their respective domains by 4b and 4(d)i in Definition 3.4.

Applying $\overline{\partial}$ componentwise, we see that $\mathcal{Z}^\tau_\zeta(c)$ is a holomorphic mapping (from $B_\zeta(4\widetilde{\tau}+\tau/2)$ to $\mathbb{C}^n$). We have $B_\zeta(4\widetilde{\tau}+\tau/2) \Subset B_\zeta(4\widetilde{\tau}+3\tau/4)$ and (with the same argument as above we see that) $\mathcal{Z}^\tau_\zeta(c)$ extends to a holomorphic mapping on $B_\zeta(4\widetilde{\tau}+3\tau/4)$, so $\mathcal{Z}^\tau_\zeta(c)$ is bounded. We conclude that $\mathcal{Z}^\tau_\zeta(c) \in \mathrm{HB}(B_\zeta(4\widetilde{\tau}+\tau/2))$. Analogously we get $\mathcal{E}^\tau_\zeta(c) \in \mathrm{HB}(A_\zeta(4\widetilde{\tau}+\tau/2))$.

Hence, for $\zeta \in \mathcal{P}$ and $\tau \in (0,\tau')$, we have defined operators

$$\begin{aligned}\mathcal{E}^\tau_\zeta &: \mathrm{HB}(C_\zeta(4\widetilde{\tau}+\tau)) \to \mathrm{HB}(A_\zeta(4\widetilde{\tau}+\tau/2)),\\ \mathcal{Z}^\tau_\zeta &: \mathrm{HB}(C_\zeta(4\widetilde{\tau}+\tau)) \to \mathrm{HB}(B_\zeta(4\widetilde{\tau}+\tau/2)).\end{aligned}$$

It remains to check that those operators have all the desired properties:

Property 1 is clear and linearity in Property 2 is clear as well. Set $M_3 := 1+nCK'$, where C and K' are as in Definition 3.4. We prove the estimate in Property 2:

$$\begin{aligned}\left\|\mathcal{Z}^\tau_\zeta(c)\right\|_{B_\zeta(4\widetilde{\tau}+\tau/2)} &\leq \|\chi(\cdot,\zeta,\tau)\cdot c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + \|g\|_{B_\zeta(4\widetilde{\tau}+\tau/2)}\\ &\leq \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + \|g\|_{D_\zeta(4\widetilde{\tau}+3\tau/4)}\\ &\leq \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + \sum_{j=1}^n |g_j|_{D_\zeta(4\widetilde{\tau}+3\tau/4)}\\ &= \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + \sum_{j=1}^n \left|S^{\zeta,\frac{3}{4}\tau}(f_j)\right|_{D_\zeta(4\widetilde{\tau}+3\tau/4)}\\ &\leq \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + \sum_{j=1}^n \left|S^{\zeta,\frac{3}{4}\tau}(f_j)\right|_{1/2,D_\zeta(4\widetilde{\tau}+3\tau/4)}\\ &\leq \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + \sum_{j=1}^n C\,|f_j|_{\overline{D_\zeta(4\widetilde{\tau}+3\tau/4)}}\\ &\leq \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + C\sum_{j=1}^n |f_j|_{D_\zeta(4\widetilde{\tau}+\tau)}\\ &= \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + C\sum_{j=1}^n |f_j|_{C_\zeta(4\widetilde{\tau}+\tau)}\\ &= \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + C\sum_{j=1}^n \left|\overline{\partial}\Big(\chi(\cdot,\zeta,\tau)\cdot c_j\Big)\right|_{C_\zeta(4\widetilde{\tau}+\tau)}\end{aligned}$$

$$
\begin{aligned}
&= \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + C\sum_{j=1}^{n} \left| c_j \cdot \overline{\partial}\Big(\chi(\cdot,\zeta,\tau)\Big) \right|_{C_\zeta(4\widetilde{\tau}+\tau)} \\
&\leq \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} \\
&\quad + C\sum_{j=1}^{n} |c_j|_{C_\zeta(4\widetilde{\tau}+\tau)} \cdot \left| \overline{\partial}\Big(\chi(\cdot,\zeta,\tau)\Big) \right|_{C_\zeta(4\widetilde{\tau}+\tau)} \\
&\leq \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} \\
&\quad + C\sum_{j=1}^{n} ||c||_{C_\zeta(4\widetilde{\tau}+\tau)} \cdot \left| \overline{\partial}\Big(\chi(\cdot,\zeta,\tau)\Big) \right|_{C_\zeta(4\widetilde{\tau}+\tau)} \\
&= \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} \\
&\quad + n \cdot C \cdot ||c||_{C_\zeta(4\widetilde{\tau}+\tau)} \cdot \left| \overline{\partial}\Big(\chi(\cdot,\zeta,\tau)\Big) \right|_{C_\zeta(4\widetilde{\tau}+\tau)} \\
&\leq \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} + n \cdot C \cdot ||c||_{C_\zeta(4\widetilde{\tau}+\tau)} \cdot K' \\
&= (1 + n \cdot C \cdot K') \cdot \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)} \\
&= M_3 \cdot \|c\|_{C_\zeta(4\widetilde{\tau}+\tau)},
\end{aligned}
$$

and a similar calculation works for $\mathcal{E}_\zeta^\tau$. Note in particular that the constant M_3 does not depend on c, τ and ζ.

It remains to check Property 3. Fix $\tau \in (0, \tau')$ and let c, a and b be as in the formulation of Property 3 (note that we assigned the variable names τ and c before; we reassign them to keep notation as intuitive as possible).

For all $j \in \{1, \dots, n\}$ and $\zeta \in \mathcal{P}$ we define a $(0,1)$-form $f_{j,\zeta}$ on $D_\zeta(4\widetilde{\tau} + \tau)$ analogously to above, i.e.

$$
f_{j,\zeta} := \begin{cases} \overline{\partial}\Big(\chi(\cdot,\zeta,\tau) \cdot c(\cdot,\zeta)_j\Big) & \text{on } C_\zeta(4\widetilde{\tau}+\tau), \\ 0 = \sum_{k=1}^n 0 \mathrm{d}\bar{z}_k & \text{on } D_\zeta(4\widetilde{\tau}+\tau) \setminus C_\zeta(4\widetilde{\tau}+\tau), \end{cases}
$$

and define

$$
g_{j,\zeta} := S^{\zeta, \frac{3}{4}\tau}(f_{j,\zeta}).
$$

Analogously to above we set for all $\zeta \in \mathcal{P}$:

$$
g_\zeta := \begin{pmatrix} g_{1,\zeta} \\ \vdots \\ g_{n,\zeta} \end{pmatrix}.
$$

We write on $D_\zeta(4\widetilde{\tau} + \tau)$:

$$
f_{j,\zeta} = \sum_{k=1}^{n} h_{j,k}^{(\zeta)} \mathrm{d}\bar{z}_k,
$$

i.e. we have for all $j, k \in \{1, \ldots, n\}$:

$$h_{j,k}^{(\zeta)} = \begin{cases} c(\cdot,\zeta)_j \cdot \frac{\partial(\chi(\cdot,\zeta,\tau))}{\partial \bar{z}_k} & \text{on } C_\zeta(4\widetilde{\tau}+\tau), \\ 0 & \text{on } D_\zeta(4\widetilde{\tau}+\tau) \setminus C_\zeta(4\widetilde{\tau}+\tau). \end{cases}$$

We define the map $h_{j,k} \colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in \overline{D_\zeta(4\widetilde{\tau}+3\tau/4)}\} \to \mathbb{C}$ by

$$(z,\zeta) \mapsto h_{j,k}^{(\zeta)}(z).$$

$h_{j,k}$ is continuous by 4(d)iv in Definition 3.4. Hence we can use 4(c)iv in Definition 3.4 to deduce that the map

$$\mathcal{G}_j \colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in D_\zeta(4\widetilde{\tau}+3\tau/4)\} \to \mathbb{C}$$

defined by

$$(z,\zeta) \mapsto g_{j,\zeta}(z)$$

is continuous. Then of course the map

$$\mathcal{G} \colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in D_\zeta(4\widetilde{\tau}+3\tau/4)\} \to \mathbb{C}^n$$

defined by

$$(z,\zeta) \mapsto g_\zeta(z)$$

is continuous as well. Continuity of a and b then follows from 4(d)v in Definition 3.4. $\square$

We now give a Lipschitz estimate for holomomorphic maps depending on a parameter. The point of the matter is that this estimate will *not* depend on the parameter, which will be essential for establishing the continuous dependence in our splitting lemma on varying domains. The notation might be a bit confusing at first glance, but it will be quite convenient when we apply this lemma later.

LEMMA 3.15. *Let* $(\{(A_\zeta, B_\zeta)\}_{\zeta\in\mathcal{P}}, \widetilde{\tau})$ *be pleasant.*

If $0 < r' < r$ *and* $s > 0$ *and if* $\mathcal{F} \colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in C_\zeta(r)\} \to \mathbb{C}^n$ *is a map such that*

- $\mathcal{F}(\cdot,\zeta)$ *is holomorphic on* $C_\zeta(r)$ *for all* $\zeta \in \mathcal{P}$,
- $\mathrm{dist}_{C_\zeta(r)}(\mathcal{F}(\cdot,\zeta), \mathrm{Id}) < s$ *for all* $\zeta \in \mathcal{P}$,

then we have for all $\zeta \in \mathcal{P}$ *and all* $x, y \in \mathbb{C}^n$ *with* $\{lx+(1-l)y \colon l \in [0,1]\} \subseteq C_\zeta(r')$*:*

$$\|\mathcal{F}(y,\zeta) - \mathcal{F}(x,\zeta)\| \le \left(1 + 4n^{5/2} \cdot \frac{s}{r-r'}\right) \cdot \|y-x\|.$$

Proof. Let r', r, s and $\mathcal{F}$ be as above. Let $\zeta \in \mathcal{P}$ and assume $x, y \in \mathbb{C}^n$ satisfy $\{lx + (1-l)y \colon l \in [0,1]\} \subseteq C_\zeta(r')$. Since the map $\mathcal{F}(\cdot,\zeta) - \mathrm{Id}$ is bounded and holomorphic on $C_\zeta(r)$, we can apply Lemma 3.9 to deduce that

$$\begin{aligned}\big\|\big(\mathcal{F}(\cdot,\zeta)-\mathrm{Id}\big)(y)-\big(\mathcal{F}(\cdot,\zeta)-\mathrm{Id}\big)(x)\big\| &\le 4n^{5/2}\cdot\frac{\|\mathcal{F}(\cdot,\zeta)-\mathrm{Id}\|_{C_\zeta(r)}}{r-r'}\cdot\|y-x\|\\ &\le 4n^{5/2}\cdot\frac{s}{r-r'}\cdot\|y-x\|.\end{aligned}$$

Using this we calculate:

$$\begin{aligned}\|\mathcal{F}(y,\zeta)-\mathcal{F}(x,\zeta)\| &\le \|\mathcal{F}(y,\zeta)-\mathcal{F}(x,\zeta)-y+x\| + \|y-x\|\\ &= \big\|\big(\mathcal{F}(\cdot,\zeta)-\mathrm{Id}\big)(y)-\big(\mathcal{F}(\cdot,\zeta)-\mathrm{Id}\big)(x)\big\| + \|y-x\|\\ &\le \left(1+4n^{5/2}\cdot\frac{s}{r-r'}\right)\cdot\|y-x\|,\end{aligned}$$

and we are done. □

Remark. In the proof of Lemma 3.15, the fact that $(\{(A_\zeta,B_\zeta)\}_{\zeta\in\mathcal{P}},\widetilde{\tau})$ is pleasant is only used to deduce that the sets C_ζ are (welldefined and) nonempty, so that the sets $C_\zeta(r)$ (resp. $C_\zeta(r')$) are welldefined, nonempty and open subsets of $\mathbb{C}^n$. Note furthermore that there is no continuity assumption on the map $\mathcal{F}$ in Lemma 3.15.

The following lemma is the key ingredient for the proof of our splitting lemma on varying domains; it is based on [**4**, Lemma 8.7.7 on p. 363]. We use the additive splitting obtained from Lemma 3.14 to construct maps which in some sense are "close" to giving a compositional splitting. In the proof of Theorem 3.6 we will repeatedly apply this (while shrinking the occuring domains in a controlled way) to obtain a compositional splitting in the limit.

Lemma 3.16. *If $(\{(A_\zeta,B_\zeta)\}_{\zeta\in\mathcal{P}},\widetilde{\tau})$ is pleasant then there exist constants $r_0 > 0$ and $M_4, M_5 > 1$ satisfying the following:*

If $0 < r \le r_0$ and if $\gamma\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(4\widetilde{\tau}+r)\}\to\mathbb{C}^n$ is a continuous mapping such that $\gamma(\cdot,\zeta)$ is an injective holomophic map on $C_\zeta(4\widetilde{\tau}+r)$ with $\mathrm{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id}) < r/(16M_4)$ for all $\zeta\in\mathcal{P}$, then there exist mappings

$$\alpha\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in A_\zeta(4\widetilde{\tau}+r/2)\}\to\mathbb{C}^n$$
$$\beta\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in B_\zeta(4\widetilde{\tau}+r/2)\}\to\mathbb{C}^n$$

such that:

(1) α and β are continuous.
(2) $\alpha(\cdot,\zeta)$ (resp. $\beta(\cdot,\zeta)$) is an injective holomorphic map on $A_\zeta(4\widetilde{\tau}+r/4)$ (resp. $B_\zeta(4\widetilde{\tau}+r/4)$) for all $\zeta\in\mathcal{P}$.

(3) *For all $\zeta \in \mathcal{P}$ we have the following estimates:*

$$\operatorname{dist}_{A_\zeta(4\widetilde{\tau}+r/2)}(\alpha(\cdot,\zeta),\mathrm{Id}) \le M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id}),$$
$$\operatorname{dist}_{B_\zeta(4\widetilde{\tau}+r/2)}(\beta(\cdot,\zeta),\mathrm{Id}) \le M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id}),$$

where M_3 is as in Lemma 3.14.

(4) *The mapping $\widetilde{\gamma}\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in C_\zeta(4\widetilde{\tau}+r/8)\} \to \mathbb{C}^n$ defined by*

$$(z,\zeta) \mapsto \Big(\big(\beta(\cdot,\zeta)\big)^{-1} \circ \gamma(\cdot,\zeta) \circ \alpha(\cdot,\zeta)\Big)(z)$$

is (welldefined and) continuous and for all $\zeta \in \mathcal{P}$ the map $\widetilde{\gamma}(\cdot,\zeta)\colon C_\zeta(4\widetilde{\tau}+r/8) \to \mathbb{C}^n$ is injective and holomorphic. Furthermore we have the following estimate for all $\zeta \in \mathcal{P}$:

$$\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r/8)}(\widetilde{\gamma}(\cdot,\zeta),\mathrm{Id}) \le M_5 \cdot \frac{1}{r} \cdot \big(\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\big)^2.$$

NOTE 3.17. $\beta(\cdot,\zeta)$ is not necessarily injective in the situation of Property 4 in Lemma 3.16, but (as we'll show when proving Property 2) it's injective on $B_\zeta(4\widetilde{\tau}+r/4)$. So, whenever we write $\big(\beta(\cdot,\zeta)\big)^{-1}$, we actually mean $\big(\beta(\cdot,\zeta)|_{B_\zeta(4\widetilde{\tau}+r/4)}\big)^{-1}$ (which makes sense after restricting the range to $\big(\beta(\cdot,\zeta)\big)(B_\zeta(4\widetilde{\tau}+r/4))$).

PROOF OF LEMMA 3.16. Let M_3 be as in Lemma 3.14, let K be as in Lemma 3.7, let τ' be as in Definition 3.4 and let M_2 be as in Lemma 3.11. Set

$$r_0 := \frac{1}{2}\tau',$$
$$M_4 := 2 \cdot \max\left\{2^{11}M_3, \frac{M_3}{4K}\right\},$$
$$M_5 := 32 M_2 M_3^2.$$

Let $0 < r \le r_0$ and let $\gamma\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in C_\zeta(4\widetilde{\tau}+r)\} \to \mathbb{C}^n$ be a continuous mapping such that $\gamma(\cdot,\zeta)$ is an injective holomophic map on $C_\zeta(4\widetilde{\tau}+r)$ with $\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id}) < r/(16M_4)$ for all $\zeta \in \mathcal{P}$.

Define $c\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in C_\zeta(4\widetilde{\tau}+r)\} \to \mathbb{C}^n$ by

$$(z,\zeta) \mapsto \gamma(z,\zeta) - z.$$

$c(\cdot,\zeta)$ is holomorphic and bounded by $r/(16M_4)$ for all $\zeta \in \mathcal{P}$, i.e. we have

$$c(\cdot,\zeta) \in \mathrm{HB}(C_\zeta(4\widetilde{\tau}+r))$$

for all $\zeta \in \mathcal{P}$. Let $\mathcal{E}_\zeta^\tau$ and $\mathcal{Z}_\zeta^\tau$ be as in Lemma 3.14 for all $\zeta \in \mathcal{P}, \tau \in (0,\tau')$. Then, by Lemma 3.14, the maps

$$a\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in A_\zeta(4\widetilde{\tau}+r/2)\} \to \mathbb{C}^n,$$
$$(z,\zeta) \mapsto \big(\mathcal{E}_\zeta^r(c(\cdot,\zeta))\big)(z),$$

$$\begin{aligned} b\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in B_\zeta(4\widetilde{\tau} + r/2)\} &\to \mathbb{C}^n, \\ (z,\zeta) &\mapsto \big(\mathcal{Z}_\zeta^r(c(\cdot,\zeta))\big)(z) \end{aligned}$$

are welldefined and continuous. We define

$$\begin{aligned} \alpha\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in A_\zeta(4\widetilde{\tau} + r/2)\} &\to \mathbb{C}^n, \\ (z,\zeta) &\mapsto z + a(z,\zeta), \\ \beta\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in B_\zeta(4\widetilde{\tau} + r/2)\} &\to \mathbb{C}^n, \\ (z,\zeta) &\mapsto z + b(z,\zeta). \end{aligned}$$

We have to prove that α and β have the desired properties. Property 1 is clear by continuity of a and b. We will check Properties 2 and 3 for α; for β they will follow analogously.

$\alpha(\cdot,\zeta)$ is holomorphic on $A_\zeta(4\widetilde{\tau}+r/2)$, since $a(\cdot,\zeta)$ is holomorphic on $A_\zeta(4\widetilde{\tau}+r/2)$. Furthermore we have by Lemma 3.14:

$$\begin{aligned} \operatorname{dist}_{A_\zeta(4\widetilde{\tau}+r/2)}(\alpha(\cdot,\zeta),\mathrm{Id}) &= \|a(\cdot,\zeta)\|_{A_\zeta(4\widetilde{\tau}+r/2)} \\ &\leq M_3 \cdot \|c(\cdot,\zeta)\|_{C_\zeta(4\widetilde{\tau}+r)} \\ &= M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id}). \end{aligned}$$

This shows Property 3. We still have to prove injectivity in Property 2. We have, since $\big(A_\zeta(4\widetilde{\tau}+r/4)\big)(r/4) = A_\zeta(4\widetilde{\tau}+r/2)$:

$$\begin{aligned} \|a(\cdot,\zeta)\|_{\left(A_\zeta(4\widetilde{\tau}+r/4)\right)(r/4)} &= \|a(\cdot,\zeta)\|_{A_\zeta(4\widetilde{\tau}+r/2)} \\ &\leq M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id}) \\ &< M_3 \cdot \frac{r}{16M_4} \\ &= \frac{M_3}{4M_4} \cdot \frac{r}{4} \\ &< \frac{M_3}{4 \cdot \frac{M_3}{4K}} \cdot \frac{r}{4} \\ &= K \cdot \frac{r}{4}. \end{aligned}$$

Hence we can apply Lemma 3.7 to deduce that $\alpha(\cdot,\zeta)$ is injective on $A_\zeta(4\widetilde{\tau}+r/4)$, which proves Property 2. Note that $\alpha(\cdot,\zeta)$ is holomorphic on the bigger domain $A_\zeta(4\widetilde{\tau}+r/2)$, but not necessarily injective there. Compare this to Remark 3.8. The existence of the constant K ensures that M_4 does not depend on ζ and r.

It remains to check Property 4. $\alpha(\cdot,\zeta)$, $\beta(\cdot,\zeta)$ and $\gamma(\cdot,\zeta)$ are injective and holomorphic on $C_\zeta(4\widetilde{\tau}+r/4) = \big(C_\zeta(4\widetilde{\tau}+r/8)\big)(r/8)$. Assume first that $\gamma(\cdot,\zeta)$ is not the identity on $C_\zeta(4\widetilde{\tau}+r)$. Then we have:

$$\operatorname{dist}_{\left(C_\zeta(4\widetilde{\tau}+r/8)\right)(r/8)}(\gamma(\cdot,\zeta),\mathrm{Id}) \leq \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})$$

$$
\begin{aligned}
&\le M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\\
&< 2M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\\
&< 2M_3 \cdot \frac{r}{16M_4}\\
&= \frac{M_3}{M_4}\cdot\frac{r}{8}\\
&< \frac{1}{4}\cdot\frac{r}{8}
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{dist}_{\big(C_\zeta(4\widetilde{\tau}+r/8)\big)(r/8)}(\alpha(\cdot,\zeta),\mathrm{Id}) &\le \operatorname{dist}_{A_\zeta(4\widetilde{\tau}+r/2)}(\alpha(\cdot,\zeta),\mathrm{Id})\\
&= \|a(\cdot,\zeta)\|_{A_\zeta(4\widetilde{\tau}+r/2)}\\
&\le M_3\cdot\|c(\cdot,\zeta)\|_{C_\zeta(4\widetilde{\tau}+r)}\\
&= M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\\
&< 2M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\\
&< \frac{1}{4}\cdot\frac{r}{8}.
\end{aligned}
$$

Analogously we deduce:

$$
\begin{aligned}
\operatorname{dist}_{\big(C_\zeta(4\widetilde{\tau}+r/8)\big)(r/8)}(\beta(\cdot,\zeta),\mathrm{Id}) &< 2M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\\
&< \frac{1}{4}\cdot\frac{r}{8}.
\end{aligned}
$$

Then, by Lemma 3.11, the mapping

$$\widetilde{\gamma}(\cdot,\zeta)\colon C_\zeta(4\widetilde{\tau}+r/8)\to\mathbb{C}^n$$

given by

$$z\mapsto\Big(\big(\beta(\cdot,\zeta)\big)^{-1}\circ\gamma(\cdot,\zeta)\circ\alpha(\cdot,\zeta)\Big)(z)$$

is welldefined, injective and holomorphic. Since $c(\cdot,\zeta)=b(\cdot,\zeta)-a(\cdot,\zeta)$ on $C_\zeta(4\widetilde{\tau}+r/2)$ by Lemma 3.14, we can apply Lemma 3.11 to deduce the following estimate:

$$
\begin{aligned}
\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r/8)}(\widetilde{\gamma}(\cdot,\zeta),\mathrm{Id}) &\le M_2\cdot\frac{\Big(2M_3\cdot\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\Big)^2}{(r/8)}\\
&= 32M_2M_3^2\cdot\frac{1}{r}\cdot\Big(\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\Big)^2\\
&= M_5\cdot\frac{1}{r}\cdot\Big(\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})\Big)^2.
\end{aligned}
$$

Now assume that $\gamma(\cdot,\zeta)$ is the identity on $C_\zeta(4\widetilde{\tau}+r)$. The above estimates don't work in this case, since we don't get the strict inequality required for applying

Lemma 3.11. But in this case the estimates in Lemma 3.14 give us that $\alpha(\cdot,\zeta)$ and $\beta(\cdot,\zeta)$ are the identities on their respective domains. But then the mapping

$$\widetilde{\gamma}(\cdot,\zeta)\colon C_\zeta(4\widetilde{\tau}+r/8)\to\mathbb{C}^n$$

given by

$$z\mapsto\Big(\big(\beta(\cdot,\zeta)\big)^{-1}\circ\gamma(\cdot,\zeta)\circ\alpha(\cdot,\zeta)\Big)(z)$$

is the identity and hence welldefined, injective and holomorphic. The claimed estimate then reads $0\le 0$ which is trivial.

We have shown that $\widetilde{\gamma}(\cdot,\zeta)$ is welldefined for all $\zeta\in\mathcal{P}$. That implies that $\widetilde{\gamma}$ is welldefined.

The only thing left to do is to show that $\widetilde{\gamma}$ is continuous.

Remark 3.18. At first glance one might think that $\widetilde{\gamma}$ is trivially continuous, since it's the composition of continuous maps. At second glance, however, one notes that the maps we're composing depend on ζ. It's not hard to work around that, but then it's still not clear whether the map given by

$$(z,\zeta)\mapsto\big(\beta(\cdot,\zeta)\big)^{-1}(z)$$

is continuous (on some "big enough" domain). It's continuous in z, since $\beta(\cdot,\zeta)$ in injective and holomorphic on $B_\zeta(4\widetilde{\tau}+r/4)$, but that isn't enough for our purposes. One might also try to prove continuity of $\widetilde{\gamma}$ using the sequence criterion for continuity. That is not sufficient however, since $\mathcal{P}$ is not necessarily first-countable. Hence we have to work a little more to deduce continuity of $\widetilde{\gamma}$.

We'll start off by making some definitions:

$$\begin{aligned}
H_0 &:= \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(4\widetilde{\tau}+r/8)\},\\
H_1 &:= \{(z,\zeta,\zeta',\zeta'')\in\mathbb{C}^n\times\mathcal{P}\times\mathcal{P}\times\mathcal{P}\colon (z,\zeta)\in H_0,\\
&\qquad \alpha(z,\zeta)\in C_{\zeta'}(4\widetilde{\tau}+r),\gamma(\alpha(z,\zeta),\zeta')\in C_{\zeta''}(4\widetilde{\tau}+r/8+r/2^{14})\},\\
H_2 &:= \{(\widetilde{z},\zeta',\zeta'')\in\mathbb{C}^n\times\mathcal{P}\times\mathcal{P}\colon \widetilde{z}\in C_{\zeta'}(4\widetilde{\tau}+r),\\
&\qquad \gamma(\widetilde{z},\zeta')\in C_{\zeta''}(4\widetilde{\tau}+r/8+r/2^{14})\},\\
H_3 &:= \{(\widehat{z},\zeta'')\in\mathbb{C}^n\times\mathcal{P}\colon \widehat{z}\in C_{\zeta''}(4\widetilde{\tau}+r/8+r/2^{14})\}.
\end{aligned}$$

We of course assume all of them to be equipped with the respective subspace topologies. We define maps

$$\begin{aligned}
\phi_0&\colon H_0\to\mathbb{C}^n\times\mathcal{P}\times\mathcal{P}\times\mathcal{P},\\
\phi_1&\colon H_1\to\mathbb{C}^n\times\mathcal{P}\times\mathcal{P},\\
\phi_2&\colon H_2\to\mathbb{C}^n\times\mathcal{P},\\
\lambda&\colon H_3\to\mathbb{C}^n
\end{aligned}$$

as follows:

$$\begin{aligned} H_0 \ni (z,\zeta) &\overset{\phi_0}{\longmapsto} (z,\zeta,\zeta,\zeta), \\ H_1 \ni (z,\zeta,\zeta',\zeta'') &\overset{\phi_1}{\longmapsto} (\alpha(z,\zeta),\zeta',\zeta''), \\ H_2 \ni (\widetilde{z},\zeta',\zeta'') &\overset{\phi_2}{\longmapsto} (\gamma(\widetilde{z},\zeta'),\zeta''), \\ H_3 \ni (\widehat{z},\zeta'') &\overset{\lambda}{\mapsto} \big(\beta(\cdot,\zeta'')\big)^{-1}(\widehat{z}). \end{aligned}$$

Looking at how the occuring sets are defined, it's clear that ϕ_0, ϕ_1 and ϕ_2 are welldefined. We still have to check that λ is welldefined. Remember: Whenever we write $\big(\beta(\cdot,\zeta)\big)^{-1}$, we actually mean $\big(\beta(\cdot,\zeta)|_{B_\zeta(4\widetilde{\tau}+r/4)}\big)^{-1}$. We have:

$$\begin{aligned} \operatorname{dist}_{A_\zeta(4\widetilde{\tau}+r/2)}(\alpha(\cdot,\zeta),\mathrm{Id}) &= \|a(\cdot,\zeta)\|_{A_\zeta(4\widetilde{\tau}+r/2)} \\ &\le M_3 \cdot \|c(\cdot,\zeta)\|_{C_\zeta(4\widetilde{\tau}+r)} \\ &= M_3 \cdot \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id}) \\ &< M_3 \cdot \frac{r}{16M_4} \\ &\le \frac{M_3 \cdot r}{16\cdot 2\cdot 2^{11} M_3} \\ &= \frac{r}{2^{16}}. \end{aligned}$$

Analogously we get for all $\zeta \in \mathcal{P}$:

$$\begin{aligned} \operatorname{dist}_{B_\zeta(4\widetilde{\tau}+r/2)}(\beta(\cdot,\zeta),\mathrm{Id}) &< \frac{r}{2^{16}}, \\ \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id}) &< \frac{r}{2^{16}}. \end{aligned}$$

Hence we can use Lemma 3.12 to deduce that $\big(\beta(\cdot,\zeta)\big)(B_\zeta(4\widetilde{\tau}+r/4))$ contains $\big(B_\zeta(4\widetilde{\tau})\big)(r/4 - r/2^{16}) = B_\zeta(4\widetilde{\tau}+r/4-r/2^{16})$ for all $\zeta \in \mathcal{P}$, i.e. $\big(\beta(\cdot,\zeta)\big)^{-1}$ is welldefined on $C_\zeta(4\widetilde{\tau}+r/8+r/2^{14}) \subseteq B_\zeta(4\widetilde{\tau}+r/4-r/2^{16})$. This shows that λ is welldefined.

We have for all $\zeta \in \mathcal{P}$:

$$\begin{aligned} \operatorname{dist}_{B_\zeta(4\widetilde{\tau}+r/4-r/2^{16})}\Big(\big(\beta(\cdot,\zeta)\big)^{-1},\mathrm{Id}\Big) &\le \operatorname{dist}_{B_\zeta(4\widetilde{\tau}+r/2)}(\beta(\cdot,\zeta),\mathrm{Id}) \\ &< \frac{r}{2^{16}}. \end{aligned} \tag{*1}$$

Since both $\operatorname{dist}_{A_\zeta(4\widetilde{\tau}+r/2)}(\alpha(\cdot,\zeta),\mathrm{Id})$ and $\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+r)}(\gamma(\cdot,\zeta),\mathrm{Id})$ are smaller than $r/2^{16}$ for all $\zeta \in \mathcal{P}$, we get:

$$\phi_0(H_0) \subseteq H_1.$$

Moreover, we trivially have:

$$\phi_1(H_1) \subseteq H_2,$$

$$\phi_2(H_2) \subseteq H_3.$$

Hence the composition

$$\lambda \circ \phi_2 \circ \phi_1 \circ \phi_0 \colon H_0 \to \mathbb{C}^n$$

is welldefined. We want to show that it equals $\widetilde{\gamma}\colon H_0 \to \mathbb{C}^n$. To this end let $(z,\zeta) \in H_0$. We calculate:

$$\begin{aligned}
(\lambda \circ \phi_2 \circ \phi_1 \circ \phi_0)(z,\zeta) &= (\lambda \circ \phi_2 \circ \phi_1)(z,\zeta,\zeta,\zeta)\\
&= (\lambda \circ \phi_2)(\alpha(z,\zeta),\zeta,\zeta)\\
&= \lambda\Big(\gamma\big(\alpha(z,\zeta),\zeta\big),\zeta\Big)\\
&= \big(\beta(\cdot,\zeta)\big)^{-1}\big(\gamma\big(\alpha(z,\zeta),\zeta\big)\big)\\
&= \Big(\big(\beta(\cdot,\zeta)\big)^{-1} \circ \gamma(\cdot,\zeta) \circ \alpha(\cdot,\zeta)\Big)(z)\\
&= \widetilde{\gamma}(z,\zeta).
\end{aligned}$$

Hence we have:

$$\widetilde{\gamma} = \lambda \circ \phi_2 \circ \phi_1 \circ \phi_0.$$

ϕ_0, ϕ_1 and ϕ_2 are continuous, since their component functions are continuous. If we can prove that λ is continuous, then $\widetilde{\gamma}$ will be a composition of continuous mappings and hence continuous; so it suffices to prove that λ is continuous.

Analogously to above we immediately get that the map

$$\mathcal{F}\colon \{(z',\zeta') \in \mathbb{C}^n \times \mathcal{P}\colon z' \in C_{\zeta'}(4\widetilde{\tau} + r/4 - r/2^{16})\} \to \mathbb{C}^n$$

defined by

$$(z',\zeta') \mapsto \big(\beta(\cdot,\zeta')\big)^{-1}(z')$$

is welldefined and that $\mathcal{F}(\cdot,\zeta')$ is holomorphic on $C_{\zeta'}(4\widetilde{\tau} + r/4 - r/2^{16})$ for all $\zeta' \in \mathcal{P}$. We define:

$$\begin{aligned}
L(r) &:= 1 + 4n^{5/2} \cdot \frac{r/2^{16}}{(4\widetilde{\tau} + r/4 - r/2^{16}) - (4\widetilde{\tau} + r/8 + r/2^{10})}\\
&= 1 + 4n^{5/2} \cdot \frac{1/2^{16}}{1/8 - 1/2^{10} - 1/2^{16}}\\
&> 1.
\end{aligned}$$

If $\zeta' \in \mathcal{P}$, $y_1 \in C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{12})$ and $y_2 \in \mathbb{C}^n$ with $\|y_2 - y_1\| < r/2^{10} - r/2^{12}$, then $\mathcal{F}(y_1,\zeta')$ and $\mathcal{F}(y_2,\zeta')$ are welldefined and satisfy (by (*1) and by Lemma 3.15):

$$\|\mathcal{F}(y_2,\zeta') - \mathcal{F}(y_1,\zeta')\| \le L(r) \cdot \|y_2 - y_1\|, \tag{*2}$$

(since the real line segment connecting y_1 and y_2 lies in $C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{10})$). Note that $L(r)$ does *not* depend on ζ'. It also doesn't depend on r, but (having possible future generalizations in mind) we keep notation as it is.

We want to prove that λ is continuous. To this end let $(z, \zeta) \in H_3$ (note that we assigned the variable names z and ζ before; we reassign them to keep notation as intuitive as possible).

Let $\epsilon > 0$. We have to find a neighborhood N of (z, ζ) in H_3 that satisfies $\lambda(N) \subseteq B^{(n)}(\lambda(z, \zeta), \epsilon)$, where $B^{(n)}(\lambda(z, \zeta), \epsilon)$ denotes the open ball of radius ϵ around $\lambda(z, \zeta)$ in $\mathbb{C}^n$ with respect to the euclidean metric.

$\big(\beta(\cdot, \zeta)\big)^{-1}$ is holomorphic (and hence continuous) on an open subset of $\mathbb{C}^n$ containing z, so there exists an open subset N_1 of $\mathbb{C}^n$ such that:

- $z \in N_1$,
- $N_1 \times \{\zeta\} \subseteq H_3$,
- for all $\widetilde{z} \in N_1$ we have $\left\| \big(\beta(\cdot, \zeta)\big)^{-1}(z) - \big(\beta(\cdot, \zeta)\big)^{-1}(\widetilde{z}) \right\| < \epsilon/2$.

Pick $\eta > 0$ with $\eta < \max\{\epsilon/2, r/2^{15}\}$.

If $(z', \zeta') \in H_3 \cap (N_1 \times \mathcal{P})$, then $\big(\beta(\cdot, \zeta)\big)^{-1}(z')$ is welldefined (since $N_1 \times \{\zeta\} \subseteq H_3$) and $z' \in C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{14})$ (since $(z', \zeta') \in H_3$), and hence we have by (*1):

$$\begin{aligned} B^{(n)}\left(\big(\beta(\cdot, \zeta)\big)^{-1}(z'), \eta\right) &\subseteq \Big(\big(C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{14})\big)(r/2^{16})\Big)(\eta) \\ &= C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{14} + r/2^{16} + \eta) \\ &\subseteq C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{14} + r/2^{16} + r/2^{15}) \\ &\subseteq C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{13}). \end{aligned}$$

This implies that $\beta(\cdot, \zeta')\Big(B^{(n)}\left(\big(\beta(\cdot, \zeta)\big)^{-1}(z'), \eta\right)\Big)$ is welldefined and satisfies:

$$\begin{aligned} \beta(\cdot, \zeta')\Big(B^{(n)}\left(\big(\beta(\cdot, \zeta)\big)^{-1}(z'), \eta\right)\Big) &\subseteq \Big(C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{13})\Big)(r/2^{16}) \\ &\subseteq C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{13} + r/2^{16}) \\ &\subseteq C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{12}). \end{aligned}$$

Especially we have:

$$y_{(z', \zeta')} := \beta(\cdot, \zeta')\Big(\big(\beta(\cdot, \zeta)\big)^{-1}(z')\Big) \in C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{12}).$$

$C_{\zeta'}(4\widetilde{\tau}+r/8+r/2^{13})$ is contained in $B_{\zeta'}(4\widetilde{\tau}+r/4)$, where $\beta(\cdot,\zeta')$ is invertible, i.e.

$$(*3)\qquad \big(\beta(\cdot,\zeta')\big)^{-1}\circ\beta(\cdot,\zeta') = \mathrm{Id} \text{ on } C_{\zeta'}(4\widetilde{\tau}+r/8+r/2^{13}).$$

Note 3.19. This might appear to be trivial at first glance, but remember that $\big(\beta(\cdot,\zeta')\big)^{-1}$ is just a short notation for $\big(\beta(\cdot,\zeta')|_{B_{\zeta'}(4\widetilde{\tau}+r/4)}\big)^{-1}$, while $\beta(\cdot,\zeta')$ is defined on the set $B_{\zeta'}(4\widetilde{\tau}+r/2)$. On a general subset of $B_{\zeta'}(4\widetilde{\tau}+r/2)$, the map $\big(\beta(\cdot,\zeta')\big)^{-1}\circ\beta(\cdot,\zeta')$ is not necessarily welldefined. Even if it is welldefined there, it doesn't necessarily coincide with the identity, since $\beta(\cdot,\zeta')$ is not necesarily injective on $B_{\zeta'}(4\widetilde{\tau}+r/2)$.

We define and calculate:

$$\begin{aligned} D_{(z',\zeta')} &:= B^{(n)}\left(y_{(z',\zeta')},\frac{\eta}{2L(r)}\right)\\ &\subseteq B^{(n)}\left(y_{(z',\zeta')},\frac{r}{2^{16}}\right)\\ &\subseteq C_{\zeta'}(4\widetilde{\tau}+r/8+r/2^{12}+r/2^{16})\\ &\subseteq C_{\zeta'}(4\widetilde{\tau}+r/8+r/2^{11}), \end{aligned}$$

so $\big(\beta(\cdot,\zeta')\big)^{-1}$ is defined on $D_{(z',\zeta')}$.

For all $y\in D_{(z',\zeta')}$ we have by (*3) and by (*2) (since $r/2^{16}<r/2^{10}-r/2^{12}$):

$$\begin{aligned} &\left\|\big(\beta(\cdot,\zeta')\big)^{-1}(y)-\big(\beta(\cdot,\zeta)\big)^{-1}(z')\right\|\\ =&\left\|\big(\beta(\cdot,\zeta')\big)^{-1}(y)-\mathrm{Id}\Big(\big(\beta(\cdot,\zeta)\big)^{-1}(z')\Big)\right\|\\ =&\left\|\big(\beta(\cdot,\zeta')\big)^{-1}(y)-\Big(\big(\beta(\cdot,\zeta')\big)^{-1}\circ\beta(\cdot,\zeta')\Big)\Big(\big(\beta(\cdot,\zeta)\big)^{-1}(z')\Big)\right\|\\ =&\left\|\big(\beta(\cdot,\zeta')\big)^{-1}(y)-\big(\beta(\cdot,\zeta')\big)^{-1}(y_{(z',\zeta')})\right\|\\ \le& L(r)\cdot\left\|y-y_{(z',\zeta')}\right\|\\ <&\frac{\eta}{2}, \end{aligned}$$

which directly implies:

$$(*4)\qquad D_{(z',\zeta')}\subseteq\beta(\cdot,\zeta')\Big(B^{(n)}\left(\big(\beta(\cdot,\zeta)\big)^{-1}(z'),\frac{\eta}{2}\right)\Big),$$

where welldefinedness of the latter set follows trivially from our calculations above. Remember that this holds for all $(z',\zeta')\in H_3\cap(N_1\times\mathcal{P})$.

Establishing (*4) was an essential part of proving that λ is continuous. We need, however, one more ingredient to construct the set N:

$\big(\beta(\cdot,\zeta)\big)^{-1}$ is holomorphic (and hence continuous) on N_1. Hence the map

$$h\colon H_3 \cap (N_1 \times \mathcal{P}) \to \mathbb{C}^n \times \mathcal{P},$$

defined by

$$(z',\zeta') \mapsto \left(\big(\beta(\cdot,\zeta)\big)^{-1}(z'),\zeta'\right),$$

is a restriction of a continuous map on $N_1 \times \mathcal{P}$ and hence continuous. Our above calculations show that the image $h\,(H_3 \cap (N_1 \times \mathcal{P}))$ of h is contained in the domain of the continuous map β, so by composing we see that the map

$$\mathcal{L}\colon H_3 \cap (N_1 \times \mathcal{P}) \to \mathbb{C}^n,$$

defined by

$$(z',\zeta') \mapsto \beta(\cdot,\zeta')\left(\big(\beta(\cdot,\zeta)\big)^{-1}(z')\right),$$

is welldefined and continuous. We have $\mathcal{L}(z,\zeta) = z$, so there exists an open subset $\widetilde{N}$ of $\mathbb{C}^n \times \mathcal{P}$ with the following properties:

- $(z,\zeta) \in \widetilde{N}$,
- $\mathcal{L}(z',\zeta') \in B^{(n)}\left(z, \frac{\eta}{4L(r)}\right)$ for all $(z',\zeta') \in \widetilde{N} \cap H_3 \cap (N_1 \times \mathcal{P})$,
- $\|z - z'\| < \frac{\eta}{4L(r)}$ for all $(z',\zeta') \in \widetilde{N} \cap H_3 \cap (N_1 \times \mathcal{P})$.

Now we finally have all the ingredients we need to prove continuity of λ. We set:

$$N := \widetilde{N} \cap H_3 \cap (N_1 \times \mathcal{P}).$$

This is a neighborhood of (z,ζ) in H_3. Let $(z',\zeta') \in N$. We have to prove that $\|\lambda(z',\zeta') - \lambda(z,\zeta)\| < \epsilon$.

We calculate (note that the estimates hold and all occuring expressions are welldefined, since $z' \in N_1$):

$$\begin{aligned}
\|\lambda(z',\zeta') - \lambda(z,\zeta)\| &= \left\|\big(\beta(\cdot,\zeta')\big)^{-1}(z') - \big(\beta(\cdot,\zeta)\big)^{-1}(z)\right\| \\
&\le \left\|\big(\beta(\cdot,\zeta')\big)^{-1}(z') - \big(\beta(\cdot,\zeta)\big)^{-1}(z')\right\| \\
&\quad + \left\|\big(\beta(\cdot,\zeta)\big)^{-1}(z') - \big(\beta(\cdot,\zeta)\big)^{-1}(z)\right\| \\
&< \left\|\big(\beta(\cdot,\zeta')\big)^{-1}(z') - \big(\beta(\cdot,\zeta)\big)^{-1}(z')\right\| + \frac{\epsilon}{2};
\end{aligned}$$

hence it suffices to prove that $\big(\beta(\cdot,\zeta')\big)^{-1}(z') \in B^{(n)}\left(\big(\beta(\cdot,\zeta)\big)^{-1}(z'),\epsilon/2\right)$.

We have $\eta < \epsilon/2$ and $B^{(n)}\left(\big(\beta(\cdot,\zeta)\big)^{-1}(z'),\eta\right) \subseteq C_{\zeta'}(4\widetilde{\tau} + r/8 + r/2^{13})$ (as seen in our above calculations), so by (*3) it suffices to prove that

$$z' \in \beta(\cdot,\zeta')\Big(B^{(n)}\left(\big(\beta(\cdot,\zeta)\big)^{-1}(z'),\eta\right)\Big);$$

hence, by (*4), it suffices to prove that $z' \in D_{(z',\zeta')}$:

Using the properties of N (resp. $\widetilde{N}$) we calculate:

$$\begin{aligned}\left\|y_{(z',\zeta')} - z'\right\| &= \left\|\beta(\cdot,\zeta')\Big(\big(\beta(\cdot,\zeta)\big)^{-1}(z')\Big) - z'\right\| \\ &\leq \left\|\beta(\cdot,\zeta')\Big(\big(\beta(\cdot,\zeta)\big)^{-1}(z')\Big) - z\right\| + \|z - z'\| \\ &= \|\mathcal{L}(z',\zeta') - z\| + \|z - z'\| \\ &< \frac{\eta}{4L(r)} + \frac{\eta}{4L(r)} \\ &= \frac{\eta}{2L(r)}.\end{aligned}$$

We deduce that $z' \in D_{(z',\zeta')}$ and we are done. $\square$

Remark 3.20. One might try to use (algebraic) topology and degree theory (as in the proof of Lemma 3.11) to prove continuity of λ in the proof of Lemma 3.16, but when taking some of the most obvious approaches in that direction, some of the assumptions required for applying the corresponding theorems are not fulfilled. One can work around some of the occuring problems, but one usually finds oneself in a situation where one needs a Lipschitz estimate like (*2) in the proof of Lemma 3.16. The Cauchy estimates were essential in establishing that Lipschitz estimate, so our argument (for continuity of λ) wouldn't work if we were presented with a family of diffeomorphisms (instead of a family of biholomorphisms) depending continuously on a parameter. One could ask whether the result (continuity of λ) still holds in that situation, but we won't follow up on this, since the answer is not significant for our purposes. It should, however, be noted that some of the more elementary attempts to construct a counterexample (e.g. where $\mathcal{P}$ is an open subset of some $\mathbb{R}^k$ and all the diffeomorphisms are defined on the same open set) are doomed to fail, since one can apply Invariance of Domain to a map given by

$$(z,\zeta) \mapsto (\beta(z,\zeta),\zeta).$$

The following result gives us that compositions are well-behaved under uniform convergence. We will use it in the proof of Theorem 3.6 to deduce that the sequences of maps we will have constructed yield a compositional splitting in the limit.

Lemma 3.21. *Let $\emptyset \neq U, V \Subset \mathbb{C}^n$ be open and let $W \Subset V$. Assume*

$$(f_m \colon U \to \mathbb{C}^n)_{m\in\mathbb{Z}_{\geq 0}},$$
$$(g_m \colon V \to \mathbb{C}^n)_{m\in\mathbb{Z}_{\geq 0}},$$

are sequences of continuous maps such that:

- *$f_m(U) \subseteq W$ for all $m \in \mathbb{Z}_{\geq 0}$,*
- *$(f_m)_{m\in\mathbb{Z}_{\geq 0}}$ converges uniformly on U to a (continuous) map $f\colon U \to \mathbb{C}^n$,*
- *$(g_m)_{m\in\mathbb{Z}_{\geq 0}}$ converges uniformly on W to a continuous map $g\colon V \to \mathbb{C}^n$.*

Then $g \circ f$ is welldefined (i.e. $f(U) \subseteq V$) and $(g_m \circ f_m)_{m\in\mathbb{Z}_{\geq 0}}$ converges uniformly on U to $g \circ f$.

Proof. Since $f_m(U) \subseteq W$ for all $m \in \mathbb{Z}_{\geq 0}$ and $W \Subset V$, we get $f(U) \Subset V$, i.e. $g \circ f$ is welldefined. Pick $V' \Subset V$, such that $f(U) \Subset V'$ and $W \Subset V'$.

Let $\epsilon > 0$. Choose $K_1 \in \mathbb{Z}_{\geq 0}$, such that

$$\|g_m - g\|_W < \frac{\epsilon}{2}$$

for all $m \geq K_1$. The closure $\overline{V'}$ of V' in $\mathbb{C}^n$ is compact and contained in V, so by the uniform continuity theorem there exists $\delta > 0$, such that

$$\|g(v_2) - g(v_1)\| < \frac{\epsilon}{2}$$

for all $v_1, v_2 \in \overline{V'}$ with $\|v_2 - v_1\| < \delta$. Choose $K_2 \in \mathbb{Z}_{\geq 0}$, such that

$$\|f_m - f\|_U < \delta$$

for all $m \geq K_2$. Set $K := \max\{K_1, K_2\}$ and let $m \in \mathbb{Z}_{\geq 0}$ with $m > K$. We calculate for some $x \in U$ (note that all occuring maps are welldefined on U):

$$\begin{aligned} \|(g \circ f)(x) - (g_m \circ f_m)(x)\| &\leq \|(g \circ f)(x) - (g \circ f_m)(x)\| \\ &\quad + \|(g \circ f_m)(x) - (g_m \circ f_m)(x)\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Hence we have $\|(g \circ f) - (g_m \circ f_m)\|_U \leq \epsilon$ and we are done. $\square$

Now we are finally ready to prove Theorem 3.6. We will repeatedly apply Lemma 3.16 to construct our compositional splitting. Continuous dependence on the parameter will follow from the estimates we'll establish along the way; we will use them to show that the maps we will have constructed are uniform limits of continuous maps.

For the sake of convenience we restate Theorem 3.6:

Theorem 3.6. *If $(\{(A_\zeta, B_\zeta)\}_{\zeta\in\mathcal{P}}, \widetilde{\tau})$ is pleasant, then for each $\eta \in \mathbb{R}_{>0}$ there exists $\epsilon_\eta \in \mathbb{R}_{>0}$ such that:*

If $\mu > 5\widetilde{\tau}$ and if $\{\gamma_\zeta\}_{\zeta\in\mathcal{P}}$ is a family of injective holomorphic maps $\gamma_\zeta\colon C_\zeta(\mu) \to \mathbb{C}^n$ satisfying

- $\gamma\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(\mu)\}\to\mathbb{C}^n$, $(z,\zeta)\mapsto\gamma_\zeta(z)$ *is continuous,*
- $\operatorname{dist}_{C_\zeta(\mu)}(\gamma_\zeta,\mathrm{Id})<\epsilon_\eta$ *for all* $\zeta\in\mathcal{P}$,

then there exist families $\{\alpha_\zeta\}_{\zeta\in\mathcal{P}}$ *and* $\{\beta_\zeta\}_{\zeta\in\mathcal{P}}$ *of injective holomorphic maps* $\alpha_\zeta\colon A_\zeta(2\widetilde{\tau})\to\mathbb{C}^n$ *and* $\beta_\zeta\colon B_\zeta(2\widetilde{\tau})\to\mathbb{C}^n$ *having the following properties:*

(1) *For all* $\zeta\in\mathcal{P}$ *we have* $\gamma_\zeta=\beta_\zeta\circ{\alpha_\zeta}^{-1}$ *on* $C_\zeta(\widetilde{\tau})$,
(2) $\operatorname{dist}_{A_\zeta(2\widetilde{\tau})}(\alpha_\zeta,\mathrm{Id})<\eta$ *and* $\operatorname{dist}_{B_\zeta(2\widetilde{\tau})}(\beta_\zeta,\mathrm{Id})<\eta$,
(3) *The maps* α *and* β *are continuous, where*

$$\alpha\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in A_\zeta(2\widetilde{\tau})\}\to\mathbb{C}^n,(z,\zeta)\mapsto\alpha_\zeta(z),$$
$$\beta\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in B_\zeta(2\widetilde{\tau})\}\to\mathbb{C}^n,(z,\zeta)\mapsto\beta_\zeta(z).$$

PROOF. Let τ' be as in Definition 3.4, let K be as in Lemma 3.7, let M_3 be as in Lemma 3.14 and let r_0, M_4 and M_5 be as in Lemma 3.16. In the proof of Lemma 3.16 we saw that M_4 can be chosen to satisfy $M_4=2\cdot\max\left\{2^{11}M_3,\frac{M_3}{4K}\right\}$. We assume this to be the case here as well. We define:

$$R_0:=\frac{1}{2}\cdot\min\left\{1,\tau',K\cdot\frac{\tau'}{4},r_0\right\}.$$

Let $\rho\colon\mathbb{R}_{>0}\times\mathbb{R}_{\geq1}\times\mathbb{R}_{\geq1}\to\mathbb{R}_{>0}$ be as in Lemma 3.10. For $\eta\in\mathbb{R}_{>0}$ we define

$$\epsilon_\eta:=\frac{1}{2}\cdot\min\{1,\rho(R_0,M_4,M_5),\rho(\eta,M_4,M_5)\}.$$

We have to check that ϵ_η has the desired property for all $\eta\in\mathbb{R}_{>0}$. To this end let $\eta>0$, let $\mu>5\widetilde{\tau}$ and let $\{\gamma_\zeta\}_{\zeta\in\mathcal{P}}$ be a family of injective holomorphic maps $\gamma_\zeta\colon C_\zeta(\mu)\to\mathbb{C}^n$ satisfying

- $\gamma\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(\mu)\}\to\mathbb{C}^n$, $(z,\zeta)\mapsto\gamma_\zeta(z)$ is continuous,
- $\operatorname{dist}_{C_\zeta(\mu)}(\gamma_\zeta,\mathrm{Id})<\epsilon_\eta$ for all $\zeta\in\mathcal{P}$.

γ_ζ is welldefined, injective and holomorphic on $C_\zeta(4\widetilde{\tau}+R_0)$, since $4\widetilde{\tau}+R_0<\mu$ by 4a in Definition 3.4. For all positive integers m we define:

$$R_m:=\frac{R_0}{2^{3m}}=\frac{R_0}{8^m}.$$

Denote the restriction of γ to $\{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(4\widetilde{\tau}+R_0)\}$ by γ_0. Since $\epsilon_\eta<\rho(R_0,M_4,M_5)$, we can apply Lemma 3.10 to the sequence $(\epsilon_\eta,0,0,\dots)$ and get $\epsilon_\eta<R_0/(16M_4)$, i.e.

$$\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_0)}(\gamma_0(\cdot,\zeta),\mathrm{Id})<\frac{R_0}{16M_4}\text{ for all }\zeta\in\mathcal{P}.$$

We have $0<R_0<r_0$ (by definition of R_0) and γ_0 is continuous, since it's a restriction of a continuous map. Hence we can apply Lemma 3.16 to R_0 and γ_0

to obtain maps

$$\alpha_0\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in A_\zeta(4\widetilde{\tau}+R_0/2)\}\to\mathbb{C}^n,$$
$$\beta_0\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in B_\zeta(4\widetilde{\tau}+R_0/2)\}\to\mathbb{C}^n,$$

having Properties 1, 2, 3 and 4 from Lemma 3.16. By Property 4 in Lemma 3.16 the map $\gamma_1\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(4\widetilde{\tau}+R_1)\}\to\mathbb{C}^n$ defined by

$$(z,\zeta)\mapsto\Big(\big(\beta_0(\cdot,\zeta)\big)^{-1}\circ\gamma_0(\cdot,\zeta)\circ\alpha_0(\cdot,\zeta)\Big)(z)$$

is (welldefined and) continuous and for all $\zeta\in\mathcal{P}$ the map $\gamma_1(\cdot,\zeta)\colon C_\zeta(4\widetilde{\tau}+R_1)\to\mathbb{C}^n$ is injective and holomorphic. Furthermore we have the following estimate for all $\zeta\in\mathcal{P}$:

$$\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_1)}(\gamma_1(\cdot,\zeta),\mathrm{Id})\leq M_5\cdot\frac{1}{R_0}\cdot\big(\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_0)}(\gamma_0(\cdot,\zeta),\mathrm{Id})\big)^2.$$

Defining

$$\epsilon_{0,\zeta}:=\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_0)}(\gamma_0(\cdot,\zeta),\mathrm{Id})<\epsilon_\eta,$$
$$\epsilon_{1,\zeta}:=\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_1)}(\gamma_1(\cdot,\zeta),\mathrm{Id}),$$

for all $\zeta\in\mathcal{P}$, the last inequality reads

$$\epsilon_{1,\zeta}\leq M_5\cdot\frac{1}{R_0}\cdot\epsilon_{0,\zeta}^2.$$

We continue our construction inductively; let $m'\in\mathbb{Z}_{\geq 1}$ and assume we have already constructed a family $\{\gamma_k\}_{k\in\{0,1,\dots,m'\}}$ of continuous maps $\gamma_k\colon\{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(4\widetilde{\tau}+R_k)\}\to\mathbb{C}^n$ and a collection $\{\epsilon_{k,\zeta}\}_{k\in\{0,1,\dots,m'\},\zeta\in\mathcal{P}}$ of non-negative real numbers with the following properties:

- For all $k\in\{0,1,\dots,m'\},\zeta\in\mathcal{P}$ the map $\gamma_k(\cdot,\zeta)\colon C_\zeta(4\widetilde{\tau}+R_k)\to\mathbb{C}^n$ is injective and holomorphic.
- For all $k\in\{0,1,\dots,m'\},\zeta\in\mathcal{P}$ we have $\epsilon_{k,\zeta}=\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_k)}(\gamma_k(\cdot,\zeta),\mathrm{Id})$.
- For all $k\in\{1,\dots,m'\},\zeta\in\mathcal{P}$ we have $\epsilon_{k,\zeta}\leq M_5\cdot\frac{1}{R_{k-1}}\cdot\epsilon_{k-1,\zeta}^2$.
- For all $k\in\{1,\dots,m'\},\zeta\in\mathcal{P}$ we have $\epsilon_{k-1,\zeta}<\frac{R_{k-1}}{16M_4}$.

We have to construct a continuous map $\gamma_{m'+1}\colon\{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(4\widetilde{\tau}+R_{m'+1})\}\to\mathbb{C}^n$ and a collection $\{\epsilon_{m'+1,\zeta}\}_{\zeta\in\mathcal{P}}$ of non-negative real numbers, such that the above properties hold for $k=m'+1$.
We have $1/R_k=2^{3k}/R_0$ for all non-negative integers k and

$$\epsilon_{0,\zeta}<\epsilon_\eta<\rho(R_0,M_4,M_5)$$

for all $\zeta\in\mathcal{P}$, so for each ζ we can apply Lemma 3.10 to the sequence

$$(\epsilon_{0,\zeta},\epsilon_{1,\zeta},\dots,\epsilon_{m',\zeta},0,0,\dots)$$

and obtain $16M_4\epsilon_{m',\zeta} < R_0/2^{3m'} = R_{m'}$. Hence we have

$$\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_{m'})}(\gamma_{m'}(\cdot,\zeta),\mathrm{Id}) < \frac{R_{m'}}{16M_4}.$$

This implies (in combination with the fact that $0 < R_{m'} < R_0 < r_0$) that we can apply Lemma 3.16 to $R_{m'}$ and $\gamma_{m'}$. Hence we obtain maps

$$\alpha_{m'}\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in A_\zeta(4\widetilde{\tau}+R_{m'}/2)\}\to\mathbb{C}^n,$$
$$\beta_{m'}\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in B_\zeta(4\widetilde{\tau}+R_{m'}/2)\}\to\mathbb{C}^n,$$

having Properties 1, 2, 3 and 4 from Lemma 3.16. By Property 4 in Lemma 3.16 the map $\gamma_{m'+1}\colon \{(z,\zeta)\in\mathbb{C}^n\times\mathcal{P}\colon z\in C_\zeta(4\widetilde{\tau}+R_{m'+1})\}\to\mathbb{C}^n$ defined by

$$(z,\zeta)\mapsto\Big(\big(\beta_{m'}(\cdot,\zeta)\big)^{-1}\circ\gamma_{m'}(\cdot,\zeta)\circ\alpha_{m'}(\cdot,\zeta)\Big)(z)$$

is (welldefined and) continuous and for all $\zeta\in\mathcal{P}$ the map $\gamma_{m'+1}(\cdot,\zeta)\colon C_\zeta(4\widetilde{\tau}+R_{m'+1})\to\mathbb{C}^n$ is injective and holomorphic. Furthermore we have the following estimate for all $\zeta\in\mathcal{P}$:

$$\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_{m'+1})}(\gamma_{m'+1}(\cdot,\zeta),\mathrm{Id}) \le M_5\cdot\frac{1}{R_{m'}}\cdot\big(\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_{m'})}(\gamma_{m'}(\cdot,\zeta),\mathrm{Id})\big)^2.$$

Defining

$$\epsilon_{m'+1,\zeta} := \operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_{m'+1})}(\gamma_{m'+1}(\cdot,\zeta),\mathrm{Id}),$$

for all $\zeta\in\mathcal{P}$, the last inequality reads

$$\epsilon_{m'+1,\zeta}\le M_5\cdot\frac{1}{R_{m'}}\cdot\epsilon_{m',\zeta}^2,$$

which completes our inductive construction. Note that in the course of the construction of $(\gamma_m)_{m\in\mathbb{Z}_{\ge0}}$, we have also constructed sequences of *continuous* maps $(\alpha_m)_{m\in\mathbb{Z}_{\ge0}}$ and $(\beta_m)_{m\in\mathbb{Z}_{\ge0}}$.

By construction we have for all $m\in\mathbb{Z}_{\ge0}$ and for all $\zeta\in\mathcal{P}$:

$$\begin{aligned}\operatorname{dist}_{C_\zeta(4\widetilde{\tau}+R_m)}(\gamma_m(\cdot,\zeta),\mathrm{Id}) = \epsilon_{m,\zeta}\le M_3\cdot\epsilon_{m,\zeta} &< \frac{M_3R_m}{16M_4}\\ &< \frac{R_m}{32}.\end{aligned}\tag{E1}$$

and, by Property 3 in Lemma 3.16, we also have:

$$\begin{aligned}\operatorname{dist}_{A_\zeta(4\widetilde{\tau}+R_m/2)}(\alpha_m(\cdot,\zeta),\mathrm{Id})\le M_3\cdot\epsilon_{m,\zeta} &< \frac{M_3R_m}{16M_4}\\ &< \frac{R_m}{32},\end{aligned}\tag{E2}$$

and

$$\text{(E3)} \qquad \begin{aligned} \operatorname{dist}_{B_\zeta(4\widetilde{\tau}+R_m/2)}(\beta_m(\cdot,\zeta),\mathrm{Id}) \le M_3 \cdot \epsilon_{m,\zeta} &< \frac{M_3 R_m}{16 M_4} \\ &< \frac{R_m}{32}. \end{aligned}$$

For all $\zeta \in \mathcal{P}, m \in \mathbb{Z}_{\ge 0}$ we define the map

$$\widetilde{\alpha}_m^{(\zeta)} \colon A_\zeta(4\widetilde{\tau} + R_m/4) \to \mathbb{C}^n$$

by:

$$z \mapsto \Big(\alpha_0(\cdot,\zeta) \circ \ldots \circ \alpha_m(\cdot,\zeta)\Big)(z).$$

Note that $\widetilde{\alpha}_m^{(\zeta)}$ is defined on $A_\zeta(4\widetilde{\tau}+R_m/4)$ and *not* on $A_\zeta(4\widetilde{\tau}+R_m/2)$. It follows inductively from (E2) and Property 2 in Lemma 3.16, that $\widetilde{\alpha}_m^{(\zeta)}$ is welldefined, injective and holomorphic.

Remark. One could try to show that $(\widetilde{\alpha}_m^{(\zeta)})_{m\in\mathbb{Z}_{\ge 0}}$ converges in sup norm on some set that doesn't depend on m. But this isn't easy to do because of the order we're composing the maps in. We'll tackle that problem by considering the inverse maps instead.

From now on, to make notation easier, we'll denote $\big(\alpha_k(\cdot,\zeta)|_{A_\zeta(4\widetilde{\tau}+R_k/4)}\big)^{-1}$ simply as $\big(\alpha_k(\cdot,\zeta)\big)^{-1}$ for all $k \in \mathbb{Z}_{\ge 0}$. Furthermore note that $\alpha_{k+1}(\cdot,\zeta)$ maps $A_\zeta(4\widetilde{\tau} + R_{k+1}/4)$ into $A_\zeta(4\widetilde{\tau} + R_k/4)$ for all $k \in \mathbb{Z}_{\ge 0}$ (by (E2)).

With that notation we have: For all $m \in \mathbb{Z}_{\ge 0}$ the map

$$\big(\widetilde{\alpha}_m^{(\zeta)}\big)^{-1} = \big(\alpha_m(\cdot,\zeta)\big)^{-1} \circ \ldots \circ \big(\alpha_0(\cdot,\zeta)\big)^{-1},$$

defined on $\big(\alpha_0(\cdot,\zeta) \circ \ldots \circ \alpha_m(\cdot,\zeta)\big)(A_\zeta(4\widetilde{\tau} + R_m/4))$, is welldefined, injective and holomorphic. We have by (E2):

$$\begin{aligned} \operatorname{dist}_{A_\zeta(4\widetilde{\tau}+R_m/4)}(\widetilde{\alpha}_m^{(\zeta)}(\cdot,\zeta),\mathrm{Id}) &\le \sum_{k=0}^{m} \operatorname{dist}_{A_\zeta(4\widetilde{\tau}+R_k/4)}(\alpha_k(\cdot,\zeta),\mathrm{Id}) \\ &< \sum_{k=0}^{m} \frac{R_k}{32} \\ &< \frac{R_0}{32} \cdot \sum_{k=0}^{\infty} \frac{1}{8^k} \\ &< \frac{R_0}{2}, \end{aligned}$$

so, by Lemma 3.12, the domain of $\big(\widetilde{\alpha}_m^{(\zeta)}\big)^{-1}$ contains $A_\zeta(4\widetilde{\tau} + R_m/4 - R_0/2)$, which in turn contains $A_\zeta(7\widetilde{\tau}/2)$ by choice of R_0. Hence, for all $m \in \mathbb{Z}_{\geq 0}$, the map $\big(\widetilde{\alpha}_m^{(\zeta)}\big)^{-1}$ is welldefined, injective and holomorphic on $A_\zeta(7\widetilde{\tau}/2)$. What we've gained from this is that the domain doesn't depend on $m \in \mathbb{Z}_{\geq 0}$.

We'll show that $\Big(\big(\widetilde{\alpha}_m^{(\zeta)}\big)^{-1}\Big)_{m\in\mathbb{Z}_{\geq 0}}$ is a Cauchy sequence with respect to the sup norm on $A_\zeta(7\widetilde{\tau}/2)$. Let $k, l \in \mathbb{Z}_{\geq 0}$ with $l > k$. We calculate, using (E2) and noting that a bijective map has the same distance to the identity as its inverse:

$$
\begin{aligned}
\Big\|\big(\widetilde{\alpha}_l^{(\zeta)}\big)^{-1} - \big(\widetilde{\alpha}_k^{(\zeta)}\big)^{-1}\Big\|_{A_\zeta(7\widetilde{\tau}/2)} &\leq \sum_{j=k}^{l-1} \Big\|\big(\widetilde{\alpha}_{j+1}^{(\zeta)}\big)^{-1} - \big(\widetilde{\alpha}_j^{(\zeta)}\big)^{-1}\Big\|_{A_\zeta(7\widetilde{\tau}/2)} \\
&= \sum_{j=k}^{l-1} \Big\|\big(\alpha_{j+1}(\cdot,\zeta)\big)^{-1} \circ \big(\widetilde{\alpha}_j^{(\zeta)}\big)^{-1} - \big(\widetilde{\alpha}_j^{(\zeta)}\big)^{-1}\Big\|_{A_\zeta(7\widetilde{\tau}/2)} \\
&= \sum_{j=k}^{l-1} \big\|\big(\alpha_{j+1}(\cdot,\zeta)\big)^{-1} - \mathrm{Id}\big\|_{\left(\widetilde{\alpha}_j^{(\zeta)}\right)^{-1}\left(A_\zeta(7\widetilde{\tau}/2)\right)} \\
&= \sum_{j=k}^{l-1} \mathrm{dist}_{\left(\widetilde{\alpha}_j^{(\zeta)}\right)^{-1}\left(A_\zeta(7\widetilde{\tau}/2)\right)} \Big(\big(\alpha_{j+1}(\cdot,\zeta)\big)^{-1}, \mathrm{Id}\Big) \\
&\leq \sum_{j=k}^{l-1} \mathrm{dist}_{\alpha_{j+1}(\cdot,\zeta)\left(A_\zeta(4\widetilde{\tau}+R_{j+1}/4)\right)} \Big(\big(\alpha_{j+1}(\cdot,\zeta)\big)^{-1}, \mathrm{Id}\Big) \\
&= \sum_{j=k}^{l-1} \mathrm{dist}_{A_\zeta(4\widetilde{\tau}+R_{j+1}/4)} \big(\alpha_{j+1}(\cdot,\zeta), \mathrm{Id}\big) \\
&\leq \sum_{j=k}^{l-1} \mathrm{dist}_{A_\zeta(4\widetilde{\tau}+R_{j+1}/2)} \big(\alpha_{j+1}(\cdot,\zeta), \mathrm{Id}\big) \\
&< \sum_{j=k}^{l-1} \frac{R_{j+1}}{32} \\
&< \frac{R_0}{32} \cdot \sum_{j=k}^{\infty} \frac{1}{8^{j+1}}.
\end{aligned}
$$

Note that all the compositions are welldefined on the respective sets. So the sequence $\Big(\big(\widetilde{\alpha}_m^{(\zeta)}\big)^{-1}\Big)_{m\in\mathbb{Z}_{\geq 0}}$ is a Cauchy sequence with respect to the sup norm on $A_\zeta(7\widetilde{\tau}/2)$ and hence converges uniformly on $A_\zeta(7\widetilde{\tau}/2)$ to a continuous map

$$\alpha_{-1}^{(\zeta)} \colon A_\zeta(7\widetilde{\tau}/2) \to \mathbb{C}^n.$$

This map is holomorphic, since its component functions are uniform limits of holomorphic functions. Analogously to above we calculate for all $k \in \mathbb{Z}_{\geq 1}$:

$$\begin{aligned}
\operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\alpha_{-1}^{(\zeta)}, \mathrm{Id}\big) &\leq \operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\alpha_{-1}^{(\zeta)}, \big(\widetilde{\alpha}_k^{(\zeta)}\big)^{-1}\big) + \operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\big(\widetilde{\alpha}_k^{(\zeta)}\big)^{-1}, \mathrm{Id}\big) \\
&\leq \operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\alpha_{-1}^{(\zeta)}, \big(\widetilde{\alpha}_k^{(\zeta)}\big)^{-1}\big) + \operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\big(\widetilde{\alpha}_0^{(\zeta)}\big)^{-1}, \mathrm{Id}\big) \\
&\quad + \sum_{j=0}^{k-1} \Big\| \big(\widetilde{\alpha}_{j+1}^{(\zeta)}\big)^{-1} - \big(\widetilde{\alpha}_j^{(\zeta)}\big)^{-1} \Big\|_{A_\zeta(7\widetilde{\tau}/2)} \\
&\leq \operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\alpha_{-1}^{(\zeta)}, \big(\widetilde{\alpha}_k^{(\zeta)}\big)^{-1}\big) + \frac{R_0}{32} \\
&\quad + \sum_{j=0}^{k-1} \frac{R_{j+1}}{32} \\
&= \operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\alpha_{-1}^{(\zeta)}, \big(\widetilde{\alpha}_k^{(\zeta)}\big)^{-1}\big) + \frac{R_0}{32} \cdot \sum_{j=0}^{k} \frac{1}{8^j} \\
&\xrightarrow{k\to\infty} 0 + \frac{R_0}{32} \cdot \frac{8}{7}.
\end{aligned}$$

Hence we have:

$$\operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\alpha_{-1}^{(\zeta)}, \mathrm{Id}\big) < \frac{R_0}{16}. \tag{E4}$$

Noting that $R_0/16 < R_0 < K \cdot \tau/4' < K \cdot \widetilde{\tau}/4$, we can apply Lemma 3.7 (to the domain $A_\zeta(13\widetilde{\tau}/4)$, the number $\widetilde{\tau}/4$ and the map $\alpha_{-1}^{(\zeta)} - \mathrm{Id}$) to deduce that the restriction of $\alpha_{-1}^{(\zeta)}$ to the domain $A_\zeta(13\widetilde{\tau}/4)$ is injective.

By (E4) and Lemma 3.12 and since $R_0 < \tau' < \widetilde{\tau}$ we get that the range of $\alpha_{-1}^{(\zeta)}|_{A_\zeta(13\widetilde{\tau}/4)}$ contains $A_\zeta(51\widetilde{\tau}/16)$. Hence the map

$$\alpha^{(\zeta)} \colon A_\zeta(51\widetilde{\tau}/16) \to \mathbb{C}^n$$

defined by

$$z \mapsto \left(\alpha_{-1}^{(\zeta)}|_{A_\zeta(13\widetilde{\tau}/4)}\right)^{-1}(z)$$

is welldefined, injective and holomorphic. Analogously we obtain maps

$$\begin{aligned}
&\widetilde{\beta}_m^{(\zeta)} \colon B_\zeta(4\widetilde{\tau} + R_m/4) \to \mathbb{C}^n \text{ for all } m \in \mathbb{Z}_{\geq 0}, \\
&\beta_{-1}^{(\zeta)} \colon B_\zeta(7\widetilde{\tau}/2) \to \mathbb{C}^n, \\
&\beta^{(\zeta)} \colon B_\zeta(51\widetilde{\tau}/16) \to \mathbb{C}^n,
\end{aligned}$$

with analogous properties. We define

$$\alpha_\zeta := \alpha^{(\zeta)}|_{A_\zeta(2\widetilde{\tau})},$$

$$\beta_\zeta := \beta^{(\zeta)}|_{B_\zeta(2\widetilde{\tau})}.$$

So we have defined families $\{\alpha_\zeta\}_{\zeta\in\mathcal{P}}$ and $\{\beta_\zeta\}_{\zeta\in\mathcal{P}}$ of injective holomorphic maps $\alpha_\zeta\colon A_\zeta(2\widetilde{\tau}) \to \mathbb{C}^n$ and $\beta_\zeta\colon B_\zeta(2\widetilde{\tau}) \to \mathbb{C}^n$. We have to check that they have Properties 1, 2 and 3 from the statement of Theorem 3.6.

For all $m \in \mathbb{Z}_{\geq 0}$ we have on $C_\zeta(4\widetilde{\tau} + R_{m+1})$ by construction:

$$\gamma_{m+1}(\cdot,\zeta) = \big(\widetilde{\beta}_m^{(\zeta)}\big)^{-1} \circ \gamma_\zeta \circ \widetilde{\alpha}_m^{(\zeta)}.$$

Compare this to Note 3.17. So, by using our distance estimates, the fact that $\big(\widetilde{\alpha}_m^{(\zeta)}\big)^{-1}$ is defined, injective and holomorphic on $A_\zeta(7\widetilde{\tau}/2)$ and that $\alpha_{-1}^{(\zeta)}$ (resp. $\beta_{-1}^{(\zeta)}$) is injective on $A_\zeta(13\widetilde{\tau}/4)$ (resp. $B_\zeta(13\widetilde{\tau}/4)$), we get:

For all $\zeta \in \mathcal{P}$ the following maps are welldefined, injective and holomorphic on $C_\zeta(25\widetilde{\tau}/8)$ for all $m \in \mathbb{Z}_{\geq 0}$:

- $\gamma_{m+1}(\cdot,\zeta) \circ \alpha_{-1}^{(\zeta)}$,
- $\gamma_{m+1}(\cdot,\zeta) \circ \big(\widetilde{\alpha}_m^{(\zeta)}\big)^{-1} = \big(\widetilde{\beta}_m^{(\zeta)}\big)^{-1} \circ \gamma_\zeta$,
- $\mathrm{Id} \circ \alpha_{-1}^{(\zeta)}$,
- $\beta_{-1}^{(\zeta)} \circ \gamma_\zeta$.

Applying Lemma 3.21 with

- $C_\zeta(25\widetilde{\tau}/8)$ in the role of U,
- $B_\zeta(7\widetilde{\tau}/2)$ in the role of V,
- $B_\zeta(13\widetilde{\tau}/4)$ in the role of W,
- $\big(\widetilde{\beta}_m^{(\zeta)}\big)^{-1}$ in the role of g_m,
- γ_ζ in the role of f_m,
- γ_ζ in the role of f,
- $\beta_{-1}^{(\zeta)}$ in the role of g,

we get:

(E5) $$\big(\widetilde{\beta}_m^{(\zeta)}\big)^{-1} \circ \gamma_\zeta \xrightarrow{m\to\infty} \beta_{-1}^{(\zeta)} \circ \gamma_\zeta \text{ uniformly on } C_\zeta(25\widetilde{\tau}/8).$$

Applying Lemma 3.21 with

- $C_\zeta(25\widetilde{\tau}/8)$ in the role of U,
- $C_\zeta(4\widetilde{\tau})$ in the role of V,
- $C_\zeta(51\widetilde{\tau}/16)$ in the role of W,
- $\gamma_{m+1}(\cdot,\zeta)$ in the role of g_m,
- $\big(\widetilde{\alpha}_m^{(\zeta)}\big)^{-1}$ in the role of f_m,
- $\alpha_{-1}^{(\zeta)}$ in the role of f,

- Id in the role of g,

we get:

$$\gamma_{m+1}(\cdot,\zeta)\circ\left(\widetilde{\alpha}_m^{(\zeta)}\right)^{-1} \xrightarrow{m\to\infty} \mathrm{Id}\circ\alpha_{-1}^{(\zeta)} \text{ uniformly on } C_\zeta(25\widetilde{\tau}/8). \tag{E6}$$

Combining (E5) and (E6) we get:

$$\beta_{-1}^{(\zeta)}\circ\gamma_\zeta=\alpha_{-1}^{(\zeta)} \text{ on } C_\zeta(25\widetilde{\tau}/8), \tag{E7}$$

since we have $\gamma_{m+1}(\cdot,\zeta)\circ\left(\widetilde{\alpha}_m^{(\zeta)}\right)^{-1}=\left(\widetilde{\beta}_m^{(\zeta)}\right)^{-1}\circ\gamma_\zeta$ for all m on that set.

We have:

$$\begin{aligned}\mathrm{dist}_{A_\zeta(2\widetilde{\tau})}(\alpha_\zeta,\mathrm{Id}) &\le \mathrm{dist}_{A_\zeta(51\widetilde{\tau}/16)}(\alpha^{(\zeta)},\mathrm{Id})\\ &\le \mathrm{dist}_{A_\zeta(13\widetilde{\tau}/4)}(\alpha_{-1}^{(\zeta)},\mathrm{Id})\\ &< \frac{R_0}{16}\\ &< \frac{\widetilde{\tau}}{16},\end{aligned} \tag{E8}$$

so, by (E8) and Lemma 3.12, the range of α_ζ contains $A_\zeta(31\widetilde{\tau}/16)$, i.e. α_ζ^{-1} is welldefined, injective and holomorphic on that set. Furthermore we have

$$\alpha_{-1}^{(\zeta)}=\alpha_\zeta^{-1} \text{ on } C_\zeta(\widetilde{\tau})$$

by definition of α_ζ. Combining this with (E7) we obtain:

$$\beta_{-1}^{(\zeta)}\circ\gamma_\zeta=\alpha_\zeta^{-1} \text{ on } C_\zeta(\widetilde{\tau}).$$

Applying (E8) and noting that ${\alpha_\zeta}^{-1}$ doesn't have bigger distance to the identity than α_ζ, we get that $\beta_\zeta\circ\alpha_\zeta^{-1}$ is welldefined, injective and holomorphic on $C_\zeta(\widetilde{\tau})$. This yields:

$$\beta_\zeta\circ\beta_{-1}^{(\zeta)}\circ\gamma_\zeta=\beta_\zeta\circ\alpha_\zeta^{-1} \text{ on } C_\zeta(\widetilde{\tau}). \tag{E9}$$

Using our distance estimates we get $\gamma_\zeta(C_\zeta(\widetilde{\tau}))\subseteq B_\zeta(13\widetilde{\tau}/4)$ and $\beta_{-1}^{(\zeta)}(\gamma_\zeta(C_\zeta(\widetilde{\tau})))\subseteq B_\zeta(51\widetilde{\tau}/16)$. The map $\beta_\zeta\circ\beta_{-1}^{(\zeta)}$ is welldefined on $\gamma_\zeta(C_\zeta(\widetilde{\tau}))$. So on $\gamma_\zeta(C_\zeta(\widetilde{\tau}))$ we have:

$$\begin{aligned}\beta_\zeta\circ\beta_{-1}^{(\zeta)} &= \beta^{(\zeta)}\circ\beta_{-1}^{(\zeta)}\\ &= \left(\beta_{-1}^{(\zeta)}|_{B_\zeta(13\widetilde{\tau}/4)}\right)^{-1}\circ\beta_{-1}^{(\zeta)}|_{B_\zeta(13\widetilde{\tau}/4)}\\ &= \mathrm{Id}.\end{aligned}$$

Compare this to Note 3.19 in order to see why this calculation was necessary. Combining this with (E9) we obtain:

$$\gamma_\zeta=\beta_\zeta\circ\alpha_\zeta^{-1} \text{ on } C_\zeta(\widetilde{\tau}),$$

which is precisely Property 1.

We'll check Property 2 now. To this end let $\zeta \in \mathcal{P}$ and define $\eta' := \min\{\eta, R_0\}$. We are only going to check the distance estimate for α_ζ; for β_ζ it follows analogously.

Consider the sequence $(\epsilon_{m,\zeta})_{m\in\mathbb{Z}_{\geq 0}}$ of non-negative real numbers constructed earlier. It satisfies:

- $\epsilon_{0,\zeta} < \epsilon_\eta < \min\{\rho(R_0, M_4, M_5), \rho(\eta, M_4, M_5)\} \leq \rho(\eta', M_4, M_5)$,
- $\epsilon_{m+1,\zeta} \leq M_5 \cdot \frac{1}{R_m} \cdot {\epsilon_{m,\zeta}}^2 = M_5 \cdot \frac{2^{3m}}{R_0} \cdot {\epsilon_{m,\zeta}}^2 \leq M_5 \cdot \frac{2^{3m}}{\eta'} \cdot {\epsilon_{m,\zeta}}^2$ for all $m \in \mathbb{Z}_{\geq 0}$.

We apply Lemma 3.10 and obtain:

$$16 M_4 \epsilon_{m,\zeta} < \frac{\eta'}{8^m} \text{ for all } m \in \mathbb{Z}_{\geq 0}.$$

Using this we calculate:

$$\begin{aligned}
\operatorname{dist}_{A_\zeta(2\widetilde{\tau})}(\alpha_\zeta, \mathrm{Id}) &\leq \operatorname{dist}_{A_\zeta(51\widetilde{\tau}/16)}(\alpha^{(\zeta)}, \mathrm{Id}) \\
&\leq \operatorname{dist}_{A_\zeta(13\widetilde{\tau}/4)}(\alpha^{(\zeta)}_{-1}, \mathrm{Id}) \\
&\leq \operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}(\alpha^{(\zeta)}_{-1}, \mathrm{Id}) \\
&\leq \limsup_{k\to\infty} \Big(\operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\alpha^{(\zeta)}_{-1}, \big(\widetilde{\alpha}^{(\zeta)}_k\big)^{-1}\big) \\
&\qquad\qquad + \operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\big(\widetilde{\alpha}^{(\zeta)}_k\big)^{-1}, \mathrm{Id}\big)\Big) \\
&\leq \limsup_{k\to\infty} \Big(\operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\alpha^{(\zeta)}_{-1}, \big(\widetilde{\alpha}^{(\zeta)}_k\big)^{-1}\big)\Big) \\
&\quad + \limsup_{k\to\infty} \Big(\operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\big(\widetilde{\alpha}^{(\zeta)}_k\big)^{-1}, \mathrm{Id}\big)\Big) \\
&= 0 + \limsup_{k\to\infty} \Big(\operatorname{dist}_{A_\zeta(7\widetilde{\tau}/2)}\big(\big(\widetilde{\alpha}^{(\zeta)}_k\big)^{-1}, \mathrm{Id}\big)\Big) \\
&\leq \limsup_{k\to\infty} \Big(\operatorname{dist}_{A_\zeta(4\widetilde{\tau}+R_k/4)}\big(\widetilde{\alpha}^{(\zeta)}_k, \mathrm{Id}\big)\Big) \\
&\leq \limsup_{k\to\infty} \left(\sum_{j=0}^{k} \operatorname{dist}_{A_\zeta(4\widetilde{\tau}+R_j/4)}\big(\alpha_j(\cdot,\zeta), \mathrm{Id}\big)\right) \\
&= \sum_{j=0}^{\infty} \operatorname{dist}_{A_\zeta(4\widetilde{\tau}+R_j/4)}\big(\alpha_j(\cdot,\zeta), \mathrm{Id}\big) \\
&\leq \sum_{j=0}^{\infty} M_3 \cdot \epsilon_{j,\zeta} \\
&\leq \eta' \cdot \frac{M_3}{16 M_4} \cdot \sum_{j=0}^{\infty} \frac{1}{8^j}
\end{aligned}$$

$$\begin{aligned}
&\leq \eta \cdot \frac{M_3}{16M_4} \cdot \frac{8}{7} \\
&< \eta \cdot \frac{1}{16} \cdot \frac{8}{7} \\
&< \eta.
\end{aligned}$$

It remains to check Property 3. We define:

$$\begin{aligned}
&\alpha \colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in A_\zeta(2\widetilde{\tau})\} \to \mathbb{C}^n, (z,\zeta) \mapsto \alpha_\zeta(z), \\
&\beta \colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in B_\zeta(2\widetilde{\tau})\} \to \mathbb{C}^n, (z,\zeta) \mapsto \beta_\zeta(z).
\end{aligned}$$

We'll show that α is continuous; continuity of β follows analogously.

For all $m \in \mathbb{Z}_{\geq 0}$ we define the map

$$\widetilde{\alpha}_m \colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in A_\zeta(4\widetilde{\tau} + R_m/4)\} \to \mathbb{C}^n,$$

by

$$(z,\zeta) \mapsto \widetilde{\alpha}_m^{(\zeta)}(z).$$

We'll show that $\widetilde{\alpha}_m$ is continuous for all $m \in \mathbb{Z}_{\geq 0}$ by induction (compare this to the first part of Remark 3.18):
$\widetilde{\alpha}_0$ is continuous, since it's a restriction of the continuous map α_0. Now assume that $\widetilde{\alpha}_k$ is continuous for some $k \in \mathbb{Z}_{\geq 0}$. We have to show that $\widetilde{\alpha}_{k+1}$ is continuous. To this end we define sets

$$\begin{aligned}
H_{0,k+1} &:= \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in A_\zeta(4\widetilde{\tau} + R_{k+1}/4)\}, \\
H_{1,k+1} &:= \{(z,\zeta,\zeta') \in \mathbb{C}^n \times \mathcal{P}^2 \colon (z,\zeta) \in H_{0,k+1}, \alpha_{k+1}(z,\zeta) \in A_{\zeta'}(4\widetilde{\tau} + R_k/4)\}, \\
H_{2,k+1} &:= \{(z',\zeta') \in \mathbb{C}^n \times \mathcal{P} \colon z' \in A_{\zeta'}(4\widetilde{\tau} + R_k/4)\},
\end{aligned}$$

and maps

$$\begin{aligned}
\phi_{0,k+1} &\colon H_{0,k+1} \to H_{1,k+1}, \\
\phi_{1,k+1} &\colon H_{1,k+1} \to H_{2,k+1},
\end{aligned}$$

given by:

$$H_{0,k+1} \ni (z,\zeta) \xmapsto{\phi_{0,k+1}} (z,\zeta,\zeta),$$

$$H_{1,k+1} \ni (z,\zeta,\zeta') \xmapsto{\phi_{1,k+1}} (\alpha_{k+1}(z,\zeta),\zeta').$$

Both $\phi_{0,k+1}$ and $\phi_{1,k+1}$ are welldefined: For $\phi_{1,k+1}$ this is clear and $\phi_{0,k+1}$ is welldefined because of our distance estimates. Both $\phi_{0,k+1}$ and $\phi_{1,k+1}$ are continuous, since their component functions are continuous (note that α_{k+1} is continuous by construction). Hence, by induction, the map

$$\widetilde{\alpha}_k \circ \phi_{1,k+1} \circ \phi_{0,k+1} \colon H_{0,k+1} \to \mathbb{C}^n,$$

is (welldefined and) continuous. A simple calculation shows

$$\widetilde{\alpha}_{k+1} = \widetilde{\alpha}_k \circ \phi_{1,k+1} \circ \phi_{0,k+1},$$

which implies that $\widetilde{\alpha}_{k+1}$ is continuous. This completes our induction.

We have shown that $\widetilde{\alpha}_m$ is continuous for all $m \in \mathbb{Z}_{\geq 0}$. If we can show that $(\widetilde{\alpha}_m)_{m \in \mathbb{Z}_{\geq 0}}$ converges to α uniformly on $\{(z, \zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in A_\zeta(2\widetilde{\tau})\}$, then continuity of α will follow from the uniform limit theorem.

Analogously to the proof of Lemma 3.15 we find an $L > 0$ (independent from $\zeta \in \mathcal{P}$) such that for all $\zeta \in \mathcal{P}$ and for all $x, y \in \mathbb{C}^n$ with $\{lx + (1-l)y \colon l \in [0,1]\} \subseteq A_\zeta(25\widetilde{\tau}/8)$ we have:

$$\left\| \alpha^{(\zeta)}(y) - \alpha^{(\zeta)}(x) \right\| \leq L \cdot \|y - x\|. \tag{E10}$$

Note that this works, since $\operatorname{dist}_{A_\zeta(51\widetilde{\tau}/16)}(\alpha^{(\zeta)}, \mathrm{Id}) < \widetilde{\tau}/16$ for all $\zeta \in \mathcal{P}$.

If $\zeta \in \mathcal{P}$ and $z \in A_\zeta(2\widetilde{\tau})$, then a simple calculation using our distance estimates shows:

- Both $(\alpha_{-1}^{(\zeta)} \circ \alpha^{(\zeta)})(z)$ and $(\alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)})(z)$ are welldefined for all $m \in \mathbb{Z}_{\geq 0}$.
- The real line segment connecting $(\alpha_{-1}^{(\zeta)} \circ \alpha^{(\zeta)})(z)$ and $(\alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)})(z)$ is contained in $A_\zeta(25\widetilde{\tau}/8)$ for all $m \in \mathbb{Z}_{\geq 0}$.

Combining this with (E10) we get:

$$\begin{aligned} &\left\| (\alpha^{(\zeta)} \circ \alpha_{-1}^{(\zeta)} \circ \alpha^{(\zeta)})(z) - (\alpha^{(\zeta)} \circ \alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)})(z) \right\| \\ \leq L \cdot &\left\| (\alpha_{-1}^{(\zeta)} \circ \alpha^{(\zeta)})(z) - (\alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)})(z) \right\|, \end{aligned}$$

for all $z \in A_\zeta(2\widetilde{\tau})$ and for all $m \in \mathbb{Z}_{\geq 0}$. Analogously to above we see (using our distance estimates) that $\alpha^{(\zeta)} \circ \alpha_{-1}^{(\zeta)} = \mathrm{Id}$ on a set containing both $\alpha^{(\zeta)}(A_\zeta(2\widetilde{\tau}))$ and $\widetilde{\alpha}_m^{(\zeta)}(A_\zeta(2\widetilde{\tau}))$ for all m. Hence, using the last inequality, we get:

$$\left\| \alpha^{(\zeta)} - \widetilde{\alpha}_m^{(\zeta)} \right\|_{A_\zeta(2\widetilde{\tau})} \leq L \cdot \left\| (\alpha_{-1}^{(\zeta)} \circ \alpha^{(\zeta)}) - (\alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)}) \right\|_{A_\zeta(2\widetilde{\tau})} \tag{E11}$$

for all m. We trivially have $\alpha_{-1}^{(\zeta)} \circ \alpha^{(\zeta)} = \mathrm{Id}$ on $A_\zeta(2\widetilde{\tau})$, so (E11) can be rewritten as:

$$\left\| \alpha^{(\zeta)} - \widetilde{\alpha}_m^{(\zeta)} \right\|_{A_\zeta(2\widetilde{\tau})} \leq L \cdot \left\| \mathrm{Id} - (\alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)}) \right\|_{A_\zeta(2\widetilde{\tau})} \text{ for all } m \in \mathbb{Z}_{\geq 0}. \tag{E12}$$

Now we're finally ready to show that $(\widetilde{\alpha}_m)_{m \in \mathbb{Z}_{\geq 0}}$ converges to α uniformly on $\mathcal{D} := \{(z, \zeta) \in \mathbb{C}^n \times \mathcal{P} \colon z \in A_\zeta(2\widetilde{\tau})\}$, which will conclude the proof of Theorem

3.6. Using (E12) we calculate for $m \in \mathbb{Z}_{\geq 1}$:

$$
\begin{aligned}
&\sup_{(z,\zeta)\in\mathcal{D}} \|\widetilde{\alpha}_m(z,\zeta) - \alpha(z,\zeta)\| \\
&\leq \sup_{\zeta\in\mathcal{P}} \left\|\widetilde{\alpha}_m^{(\zeta)} - \alpha_\zeta\right\|_{A_\zeta(2\widetilde{\tau})} \\
&= \sup_{\zeta\in\mathcal{P}} \left\|\alpha^{(\zeta)} - \widetilde{\alpha}_m^{(\zeta)}\right\|_{A_\zeta(2\widetilde{\tau})} \\
&\leq \sup_{\zeta\in\mathcal{P}} \left(L \cdot \left\| \mathrm{Id} - (\alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)}) \right\|_{A_\zeta(2\widetilde{\tau})} \right) \\
&= L \cdot \sup_{\zeta\in\mathcal{P}} \left(\left\| \mathrm{Id} - (\alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)}) \right\|_{A_\zeta(2\widetilde{\tau})} \right) \\
&\leq L \cdot \sup_{\zeta\in\mathcal{P}} \left(\limsup_{l>m, l\to\infty} \left(\left\| \alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)} - \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} \right\|_{A_\zeta(2\widetilde{\tau})} \right.\right. \\
&\qquad\qquad \left.\left. + \left\| \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} - \mathrm{Id} \right\|_{A_\zeta(2\widetilde{\tau})} \right) \right) \\
&\leq L \cdot \sup_{\zeta\in\mathcal{P}} \left(\limsup_{l>m, l\to\infty} \left(\left\| \alpha_{-1}^{(\zeta)} \circ \widetilde{\alpha}_m^{(\zeta)} - \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} \right\|_{A_\zeta(2\widetilde{\tau})} \right) \right. \\
&\qquad\qquad \left. + \limsup_{l>m, l\to\infty} \left(\left\| \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} - \mathrm{Id} \right\|_{A_\zeta(2\widetilde{\tau})} \right) \right) \\
&= L \cdot \sup_{\zeta\in\mathcal{P}} \left(\limsup_{l>m, l\to\infty} \left(\left\| \alpha_{-1}^{(\zeta)} - \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \right\|_{\widetilde{\alpha}_m^{(\zeta)}(A_\zeta(2\widetilde{\tau}))} \right) \right. \\
&\qquad\qquad \left. + \limsup_{l>m, l\to\infty} \left(\left\| \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} - \mathrm{Id} \right\|_{A_\zeta(2\widetilde{\tau})} \right) \right) \\
&\leq L \cdot \sup_{\zeta\in\mathcal{P}} \left(\limsup_{l>m, l\to\infty} \left(\left\| \alpha_{-1}^{(\zeta)} - \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \right\|_{A_\zeta(7\widetilde{\tau}/2)} \right) \right. \\
&\qquad\qquad \left. + \limsup_{l>m, l\to\infty} \left(\left\| \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} - \mathrm{Id} \right\|_{A_\zeta(2\widetilde{\tau})} \right) \right) \\
&= L \cdot \sup_{\zeta\in\mathcal{P}} \left(0 + \limsup_{l>m, l\to\infty} \left(\left\| \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} - \mathrm{Id} \right\|_{A_\zeta(2\widetilde{\tau})} \right) \right) \\
&= L \cdot \sup_{\zeta\in\mathcal{P}} \left(\limsup_{l>m, l\to\infty} \left(\left\| \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} - \left(\widetilde{\alpha}_m^{(\zeta)}\right)^{-1} \circ \widetilde{\alpha}_m^{(\zeta)} \right\|_{A_\zeta(2\widetilde{\tau})} \right) \right) \\
&= L \cdot \sup_{\zeta\in\mathcal{P}} \left(\limsup_{l>m, l\to\infty} \left(\left\| \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} - \left(\widetilde{\alpha}_m^{(\zeta)}\right)^{-1} \right\|_{\widetilde{\alpha}_m^{(\zeta)}(A_\zeta(2\widetilde{\tau}))} \right) \right) \\
&\leq L \cdot \sup_{\zeta\in\mathcal{P}} \left(\limsup_{l>m, l\to\infty} \left(\left\| \left(\widetilde{\alpha}_l^{(\zeta)}\right)^{-1} - \left(\widetilde{\alpha}_m^{(\zeta)}\right)^{-1} \right\|_{A_\zeta(7\widetilde{\tau}/2)} \right) \right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq L \cdot \sup_{\zeta\in\mathcal{P}} \left(\limsup_{l>m, l\to\infty} \left(\sum_{j=m}^{l-1} \frac{R_{j+1}}{32} \right) \right) \\
&= L \cdot \sup_{\zeta\in\mathcal{P}} \left(\frac{R_0}{32} \cdot \sum_{j=m}^{\infty} \frac{1}{8^{j+1}} \right) \\
&= \frac{LR_0}{32} \cdot \sum_{j=m}^{\infty} \frac{1}{8^{j+1}} \\
&\xrightarrow{m\to\infty} 0.
\end{aligned}
$$

Note that all the occuring compositions are welldefined on the respective sets. This concludes the proof. □

Remark 3.22. In the last part of the proof of Theorem 3.6 we showed that $\left(\widetilde{\alpha}_m\right)_m$ converges to α uniformly on $\mathcal{D}$. Because of the order of composition it was initially hard to show that $\left(\widetilde{\alpha}_m^{(\zeta)}\right)_m$ is a Cauchy sequence with respect to the sup norm (on some domain). We tackled that problem by considering the inverse maps instead and defining $\alpha^{(\zeta)}$ as the inverse of the limit. We were then able to show that $\left(\widetilde{\alpha}_m\right)_m$ converges to α uniformly on $\mathcal{D}$ with the help of a Lipschitz estimate. The Cauchy estimates were essential in establishing said Lipschitz estimate, so all the maps being holomorphic was essential for proving that the maps obtained from the compositional splitting depend continuously on the parameter. Compare this to Remark 3.20.

Remark 3.23. If $\mathcal{P}$ is a topological space, $U_\zeta \subseteq \mathbb{C}^n$ is open for all $\zeta \in \mathcal{P}$ and

$$f_m\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in U_\zeta\} \to \mathbb{C}^n,\ m = 0,1,2,3,\dots$$

are continuous maps, such that $f_m(\cdot,\zeta)$ is holomorphic on U_ζ for all m, ζ and

$$f_m(\cdot,\zeta) \xrightarrow{m\to\infty} f_\zeta \text{ uniformly on } U_\zeta$$

for all ζ, then the map

$$f\colon \{(z,\zeta) \in \mathbb{C}^n \times \mathcal{P}\colon z \in U_\zeta\} \to \mathbb{C}^n,\ (z,\zeta) \mapsto f_\zeta(z)$$

is *not* necessarily continuous; it is in fact easy to construct an example where f is not continuous with the ansatz $f_m(\cdot,\zeta) = \phi_m(\zeta) \cdot \mathrm{Id}$.

The reason we didn't encounter that problem in the proof of Theorem 3.6 is, that we used our distance estimates to find an estimate for the speed of convergence that didn't depend on ζ.

CHAPTER 4

The Main Result

Before stating the main result we need a lemma:

Lemma 4.1. *Let D be a nonempty open subset of $\mathbb{C}^n$. Assume D is bounded and has boundary of class $\mathcal{C}^1$. Then there exists $\delta_0 > 0$, such that for all $0 < \delta < \delta_0$ and for all $\zeta \in bD$ the set*

$$\{x \in \overline{D} \colon \delta \geq \|x - \zeta\| \geq \delta/2\}$$

is nonempty.

Proof. D is bounded, open and has $\mathcal{C}^1$-smooth boundary, so D only has finitely many connected components and we have

$$bD = b_{\mathbb{C}^n} D = \bigcup_{C \in \widetilde{C}} b_{\mathbb{C}^n} C,$$

where $\widetilde{C} = \{C \subseteq \mathbb{C}^n \colon C \text{ connected component of } D\}$. Hence, by a "minimum-argument", we can without loss of generality assume that D is connected.

Pick an arbitrary point $y \in D$ and pick $\delta_0 > 0$, such that $B^{(n)}(y, \delta_0) \subseteq D$. We show that $\delta_0 > 0$ is as desired:

Assume for the sake of a contradiction that there exists $p \in bD$ and $\delta \in (0, \delta_0)$, such that $\{x \in \overline{D} \colon \delta \geq \|x - p\| \geq \delta/2\} = \emptyset$. We set:

$$A := \{z \in \overline{D} \colon \|z - p\| \leq \delta\},$$
$$B := \{z \in \overline{D} \colon \|z - p\| \geq \delta/2\}.$$

A and B are both closed in $\overline{D} = A \cup B$ and $A \cap B = \{x \in \overline{D} \colon \delta \geq \|x - p\| \geq \delta/2\}$ is empty by choice of p and δ.
$\overline{D}$ is connected (since it's the closure of the connected set D), so either A or B is empty. We have $p \in A$, so we get $B = \emptyset$ and $A = \overline{D}$. We combine what we have established so far:

$$\delta_0 \leq \|y - p\| \leq \delta < \delta_0,$$

and arrive at the desired contradiction. $\square$

Remark. The assumption that D has $\mathcal{C}^1$-boundary in Lemma 4.1 is *not* unnecessary; consider for example the following set in $\mathbb{C} = \mathbb{R}^2$:

$$D = \bigcup_{m \in \mathbb{Z}_{\geq 5}} B^{(1)}\left(\frac{1}{4^m}, \frac{1}{8^m}\right),$$

where – as always – $B^{(1)}\left(\frac{1}{4^m}, \frac{1}{8^m}\right)$ denotes the open ball of radius $1/8^m$ around $1/4^m$ in $\mathbb{C}$ with respect to the euclidean metric.

1. Stating the Main Result

Let $\emptyset \neq \Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with $\mathcal{C}^2$-boundary and fix a small $\delta \in \mathbb{R}_{>0}$ (small in the sense of Lemma 4.1). For all $\zeta \in b\Omega$ we define:

$$\begin{aligned}
A_\zeta &:= \overline{\Omega} \cap \overline{B^{(n)}(\zeta, \delta)}, \\
B_\zeta &:= \left(\overline{\Omega} \cap \overline{B^{(n)}(\zeta, \delta) \setminus B^{(n)}(\zeta, \delta/2)}\right) \cup \left(\overline{\Omega} \setminus A_\zeta\right) \\
&= \overline{\Omega} \setminus B^{(n)}(\zeta, \delta/2), \\
C_\zeta &:= A_\zeta \cap B_\zeta \\
&= \overline{\Omega} \cap \overline{B^{(n)}(\zeta, \delta) \setminus B^{(n)}(\zeta, \delta/2)}, \\
D_\zeta &:= A_\zeta \cup B_\zeta.
\end{aligned}$$

Note that we have:

$$\begin{aligned}
A_\zeta &= \{z \in \overline{\Omega} \colon \|z - \zeta\| \leq \delta\}, \\
B_\zeta &= \{z \in \overline{\Omega} \colon \|z - \zeta\| \geq \delta/2\}, \\
C_\zeta &= \{z \in \overline{\Omega} \colon \delta \geq \|z - \zeta\| \geq \delta/2\}, \\
D_\zeta &= \overline{\Omega}.
\end{aligned}$$

Furthermore note that we have $C_\zeta \neq \emptyset$ for all $\zeta \in b\Omega$, since δ was chosen to be small in the sense of Lemma 4.1. We will keep this notation for the rest of this chapter. With these definitions we can now state the main result of this thesis:

Theorem 4.2. *Let $\emptyset \neq \Omega \subseteq \mathbb{C}^n$ be a bounded strictly pseudoconvex domain with $\mathcal{C}^2$-boundary and let δ be as above. If $\widetilde{\tau} > 0$ is small enough (this depends on δ), then for each $\eta \in \mathbb{R}_{>0}$ there exists $\epsilon_\eta \in \mathbb{R}_{>0}$ such that:*

If $\mu > 5\widetilde{\tau}$ and if $\{\gamma_\zeta\}_{\zeta \in b\Omega}$ is a family of injective holomorphic maps $\gamma_\zeta \colon C_\zeta(\mu) \to \mathbb{C}^n$ satisfying

- *$\gamma \colon \{(z, \zeta) \in \mathbb{C}^n \times b\Omega \colon z \in C_\zeta(\mu)\} \to \mathbb{C}^n$, $(z, \zeta) \mapsto \gamma_\zeta(z)$ is continuous,*
- *$\operatorname{dist}_{C_\zeta(\mu)}(\gamma_\zeta, \operatorname{Id}) < \epsilon_\eta$ for all $\zeta \in b\Omega$,*

then there exist families $\{\alpha_\zeta\}_{\zeta\in b\Omega}$ *and* $\{\beta_\zeta\}_{\zeta\in b\Omega}$ *of injective holomorphic maps* $\alpha_\zeta\colon A_\zeta(2\widetilde{\tau}) \to \mathbb{C}^n$ *and* $\beta_\zeta\colon B_\zeta(2\widetilde{\tau}) \to \mathbb{C}^n$ *having the following properties:*

(1) *For all* $\zeta \in b\Omega$ *we have* $\gamma_\zeta = \beta_\zeta \circ {\alpha_\zeta}^{-1}$ *on* $C_\zeta(\widetilde{\tau})$,
(2) $\operatorname{dist}_{A_\zeta(2\widetilde{\tau})}(\alpha_\zeta, \mathrm{Id}) < \eta$ *and* $\operatorname{dist}_{B_\zeta(2\widetilde{\tau})}(\beta_\zeta, \mathrm{Id}) < \eta$,
(3) *The maps* α *and* β *are continuous, where*

$$\alpha\colon \{(z,\zeta) \in \mathbb{C}^n \times b\Omega\colon z \in A_\zeta(2\widetilde{\tau})\} \to \mathbb{C}^n, (z,\zeta) \mapsto \alpha_\zeta(z),$$
$$\beta\colon \{(z,\zeta) \in \mathbb{C}^n \times b\Omega\colon z \in B_\zeta(2\widetilde{\tau})\} \to \mathbb{C}^n, (z,\zeta) \mapsto \beta_\zeta(z).$$

Note. Of course $b\Omega$ is equipped with the subspace topology it inherits from $\mathbb{C}^n$.

2. Proof of the Main Result

This section is devoted to proving the main result of this thesis, Theorem 4.2. Because of Theorem 3.6 this will reduce to checking the properties in Definition 3.4.

Proof of Theorem 4.2. By Lemma 2.26 there is an open neighborhood U of $b\Omega$, where the signed distance function ρ_Ω is $\mathcal{C}^2$-smooth and satisfies $d\rho_\Omega(x) \neq 0$ for all $x \in U$. This implies that – after making U smaller if necessary – there is an $M > 0$, such that

$$\rho := \exp\left(M \cdot \rho_\Omega\right) - 1$$

is a strictly plurisubharmonic defining function for Ω on U. Especially we have $\rho \in \mathcal{C}^2(U,\mathbb{R})$ and $d\rho(x) \neq 0$ for $x \in U$.

For all $\epsilon > 0$ we define $\rho_\epsilon\colon U \to \mathbb{R}$ by

$$z \mapsto \rho(z) - \exp\left(M \cdot \epsilon\right) + 1.$$

Furthermore we set $\Omega^{\rho_\epsilon} := (\Omega \setminus U) \cup \{z \in U\colon \rho_\epsilon(z) < 0\}$. An easy calculation shows that there exists $w_1{>}0$, such that

$$\Omega(\epsilon) = \Omega^{\rho_\epsilon} \text{ for all } 0 < \epsilon < w_1.$$

Noting that

$$\rho_\epsilon \xrightarrow{\epsilon\to 0} \rho \text{ with respect to } |\cdot|_{2,U},$$

we can apply Theorem 2.24 to find $w_2 > 0$, $C > 0$ and a collection of linear operators

$$S^{\rho_\epsilon}\colon \mathcal{C}_{0,1}(\overline{\Omega^{\rho_\epsilon}}) \to \mathcal{C}^0(\Omega^{\rho_\epsilon}),\ 0 < \epsilon < w_2,$$

as in Theorem 2.22, such that

$$|S^{\rho_\epsilon}(f)|_{1/2,\Omega^{\rho_\epsilon}} \leq C|f|_{\overline{\Omega^{\rho_\epsilon}}} \tag{F1}$$

for all $f \in \mathcal{C}_{0,1}(\overline{\Omega^{\rho_\epsilon}}) \cap \mathcal{C}^1_{0,1}(\Omega^{\rho_\epsilon})$ and for all $0 < \epsilon < w_2$. The point of the matter is that C doesn't depend on ϵ. Pick $w_3 > 0$, such that $(b\Omega)(5w_3) \Subset U$.

Let $\widetilde{\tau} > 0$ be "small enough", i.e., more precisely:

$$\widetilde{\tau} < \frac{1}{2} \cdot \min\left\{\frac{\delta}{32}, w_3, \frac{w_1}{5}, \frac{w_2}{5}\right\}.$$

Because of Theorem 3.6 it suffices to prove that $(\{(A_\zeta, B_\zeta)\}_{\zeta \in b\Omega}, \widetilde{\tau})$ is pleasant. With this in mind, we proceed to checking the properties in Definition 3.4:

From now on all the "property numbers" will refer to the property with that number *in Definition 3.4*.

Property 1 is trivially fulfilled. Property 2 is fulfilled, since δ was chosen to be small in the sense of Lemma 4.1. Note that we need the full strength of Lemma 4.1 because Ω is not necessarily connected. Compare this to Remark 2.1. Property 3 is clear, since $D_\zeta = \overline{\Omega}$ for all $\zeta \in b\Omega$.

Define $\tau' := \widetilde{\tau}/2 > 0$. Property 4a is clear. We check that Property 4b is fulfilled. For $0 < \tau < \tau'$ and $\zeta \in b\Omega$ we calculate:

$$\begin{aligned}
A_\zeta(4\widetilde{\tau}+\tau) \cap B_\zeta(4\widetilde{\tau}+\tau) &= \Big((A_\zeta \setminus B_\zeta) \cup (A_\zeta \cap B_\zeta)\Big)(4\widetilde{\tau}+\tau) \\
&\quad \cap \Big((B_\zeta \setminus A_\zeta) \cup (B_\zeta \cap A_\zeta)\Big)(4\widetilde{\tau}+\tau) \\
&= \Big((A_\zeta \setminus B_\zeta)(4\widetilde{\tau}+\tau) \cup (A_\zeta \cap B_\zeta)(4\widetilde{\tau}+\tau)\Big) \\
&\quad \cap \Big((B_\zeta \setminus A_\zeta)(4\widetilde{\tau}+\tau) \cup (B_\zeta \cap A_\zeta)(4\widetilde{\tau}+\tau)\Big) \\
&= (A_\zeta \cap B_\zeta)(4\widetilde{\tau}+\tau) \\
&\quad \cup \Big((B_\zeta \setminus A_\zeta)(4\widetilde{\tau}+\tau) \cap (A_\zeta \setminus B_\zeta)(4\widetilde{\tau}+\tau)\Big) \\
&= C_\zeta(4\widetilde{\tau}+\tau) \\
&\quad \cup \Big((B_\zeta \setminus A_\zeta)(4\widetilde{\tau}+\tau) \cap (A_\zeta \setminus B_\zeta)(4\widetilde{\tau}+\tau)\Big).
\end{aligned}$$

Hence it suffices to show that $(B_\zeta \setminus A_\zeta)(4\widetilde{\tau}+\tau) \cap (A_\zeta \setminus B_\zeta)(4\widetilde{\tau}+\tau) = \emptyset$. We have:

$$\begin{aligned}
A_\zeta \setminus B_\zeta &= \{z \in \overline{\Omega} \colon \|z - \zeta\| < \delta/2\}, \\
B_\zeta \setminus A_\zeta &= \{z \in \overline{\Omega} \colon \|z - \zeta\| > \delta\}.
\end{aligned}$$

Assume for the sake of a contradiction that there exists some

$$\omega \in (B_\zeta \setminus A_\zeta)(4\widetilde{\tau}+\tau) \cap (A_\zeta \setminus B_\zeta)(4\widetilde{\tau}+\tau).$$

Then there exist $x_1, x_2 \in \overline{\Omega}$ with $\|x_1 - \zeta\| < \delta/2$ and $\|x_2 - \zeta\| > \delta$, such that $\|\omega - x_1\| < 4\widetilde{\tau} + \tau$ and $\|\omega - x_2\| < 4\widetilde{\tau} + \tau$. We calculate:

$$\begin{aligned}
\delta &< \|x_2 - \zeta\| \\
&\leq \|x_2 - \omega\| + \|\omega - x_1\| + \|x_1 - \zeta\| \\
&< 8\widetilde{\tau} + 2\tau + \frac{\delta}{2} \\
&< 10\widetilde{\tau} + \frac{\delta}{2} \\
&< 10 \cdot \frac{\delta}{64} + \frac{\delta}{2} \\
&< \delta,
\end{aligned}$$

and arrive at the desired contradiction.

We proceed to checking Property 4c. Let $C > 0$ be as above. For $\zeta \in b\Omega$ and $0 < \tau < \tau'$ we define:

$$S^{\zeta,\tau} := S^{\rho_{4\widetilde{\tau}+\tau}}.$$

This is welldefined, since $0 < 4\widetilde{\tau} + \tau < 5\widetilde{\tau} < w_2$. Note that $S^{\zeta,\tau}$ doesn't depend on ζ (this is the case because $D_\zeta = \overline{\Omega}$ for all ζ).

Since $0 < 4\widetilde{\tau}+\tau < w_1$, the linear operator $S^{\zeta,\tau}$ maps indeed from $\mathcal{C}_{0,1}(\overline{\overline{\Omega}(4\widetilde{\tau} + \tau)})$ to $\mathcal{C}^0(\overline{\Omega}(4\widetilde{\tau} + \tau))$. Note that $\Omega(4\widetilde{\tau} + \tau) = \overline{\Omega}(4\widetilde{\tau} + \tau)$.

Since $0 < 4\widetilde{\tau} + \tau < w_1$ and $0 < 4\widetilde{\tau} + \tau < w_2$, Properties 4(c)i and 4(c)iii are clear because the operators

$$S^{\rho_\epsilon} \colon \mathcal{C}_{0,1}(\overline{\Omega^{\rho_\epsilon}}) \to \mathcal{C}^0(\Omega^{\rho_\epsilon}),\ 0 < \epsilon < w_2,$$

are "as in Theorem 2.22". Property 4(c)ii is clear from (F1). The point of the matter is that C doesn't depend on ζ and τ.

We check Property 4(c)iv; while doing so we assume $\tau \in (0, \tau')$ to be fixed. Noting that $\overline{\Omega}(4\widetilde{\tau} + \tau) = \Omega(4\widetilde{\tau} + \tau)$ we let

$$f_\zeta = \sum_{k=1}^{n} f_k^{(\zeta)} \mathrm{d}\bar{z}_k \in \mathcal{C}^1_{0,1}(\overline{\Omega(4\widetilde{\tau} + \tau)}) \text{ for all } \zeta \in b\Omega$$

and assume the function

$$f_k \colon \{(z, \zeta) \in \mathbb{C}^n \times b\Omega \colon z \in \overline{\Omega(4\widetilde{\tau} + \tau)}\} \to \mathbb{C}$$

given by

$$(z, \zeta) \mapsto f_k^{(\zeta)}(z)$$

is continuous for all $k \in \{1, \dots, n\}$. We have to show that the function

$$\mathcal{G}\colon \{(z,\zeta) \in \mathbb{C}^n \times b\Omega\colon z \in \Omega(4\widetilde{\tau}+\tau)\} \to \mathbb{C}$$

given by

$$(z,\zeta) \mapsto (S^{\zeta,\tau} f_\zeta)(z)$$

is continuous.
Since τ is fixed and $S^{\zeta,\tau}$ doesn't depend on ζ, we'll simply write S instead of $S^{\zeta,\tau}$ while checking continuity of $\mathcal{G}$. By linearity of S we can without loss of generality assume that

$$f_\zeta = f_k^{(\zeta)} \mathrm{d}\bar{z}_k \text{ for all } \zeta \in b\Omega$$

for a fixed $k \in \{1, \dots, n\}$. The function $\mathcal{G}$ is defined on the first-countable space $\Omega(4\widetilde{\tau}+\tau) \times b\Omega$, so we can check continuity using the sequence criterion for continuity. Let $((z_m,\zeta_m))_{m\in\mathbb{Z}_{\geq 0}}$ be a sequence in $\Omega(4\widetilde{\tau}+\tau) \times b\Omega$ converging to a point $(z,\zeta) \in \Omega(4\widetilde{\tau}+\tau) \times b\Omega$. We calculate:

$$\begin{aligned}
|\mathcal{G}(z,\zeta) - \mathcal{G}(z_m,\zeta_m)| \leq\ & |\mathcal{G}(z,\zeta) - \mathcal{G}(z_m,\zeta)| \\
& + |\mathcal{G}(z_m,\zeta) - \mathcal{G}(z_m,\zeta_m)| \\
=\ & \left|(S(f_k^{(\zeta)}\mathrm{d}\bar{z}_k))(z) - (S(f_k^{(\zeta)}\mathrm{d}\bar{z}_k))(z_m)\right| \\
& + |\mathcal{G}(z_m,\zeta) - \mathcal{G}(z_m,\zeta_m)|\,.
\end{aligned}$$

Since $S(f_k^{(\zeta)}\mathrm{d}\bar{z}_k) \in \mathcal{C}^0(\Omega(4\widetilde{\tau}+\tau))$, we have

$$\left|(S(f_k^{(\zeta)}\mathrm{d}\bar{z}_k))(z) - (S(f_k^{(\zeta)}\mathrm{d}\bar{z}_k))(z_m)\right| \xrightarrow{m\to\infty} 0.$$

Hence it suffices to prove that $|\mathcal{G}(z_m,\zeta) - \mathcal{G}(z_m,\zeta_m)| \xrightarrow{m\to\infty} 0$. We have by linearity of S and since $f_k^{(\zeta)}\mathrm{d}\bar{z}_k \in \mathcal{C}^1_{0,1}(\overline{\Omega(4\widetilde{\tau}+\tau)})$ and $f_k^{(\zeta_m)}\mathrm{d}\bar{z}_k \in \mathcal{C}^1_{0,1}(\overline{\Omega(4\widetilde{\tau}+\tau)})$ for all m:

$$\begin{aligned}
|\mathcal{G}(z_m,\zeta) - \mathcal{G}(z_m,\zeta_m)| &= \left|(S(f_k^{(\zeta)}\mathrm{d}\bar{z}_k))(z_m) - (S(f_k^{(\zeta_m)}\mathrm{d}\bar{z}_k))(z_m)\right| \\
&= \left|(S((f_k^{(\zeta)} - f_k^{(\zeta_m)})\mathrm{d}\bar{z}_k))(z_m)\right| \\
&\leq \left|S((f_k^{(\zeta)} - f_k^{(\zeta_m)})\mathrm{d}\bar{z}_k)\right|_{\Omega(4\widetilde{\tau}+\tau)} \\
&\leq \left|S((f_k^{(\zeta)} - f_k^{(\zeta_m)})\mathrm{d}\bar{z}_k)\right|_{1/2,\Omega(4\widetilde{\tau}+\tau)} \\
&\leq C \cdot \left|(f_k^{(\zeta)} - f_k^{(\zeta_m)})\mathrm{d}\bar{z}_k\right|_{\overline{\Omega(4\widetilde{\tau}+\tau)}}\,.
\end{aligned}$$

The map f_k is defined and continuous on the compact space $\overline{\Omega(4\widetilde{\tau}+\tau)} \times b\Omega$. Hence, with help of the uniform continuity theorem, we get:

$$\left|(f_k^{(\zeta)} - f_k^{(\zeta_m)})\mathrm{d}\bar{z}_k\right|_{\overline{\Omega(4\widetilde{\tau}+\tau)}} = |f_k(\cdot,\zeta) - f_k(\cdot,\zeta_m)|_{\overline{\Omega(4\widetilde{\tau}+\tau)}} \xrightarrow{m\to\infty} 0,$$

which implies:

$$|\mathcal{G}(z_m,\zeta)-\mathcal{G}(z_m,\zeta_m)|\xrightarrow{m\to\infty}0,$$

as desired. So we have checked Property 4c. We are going to reassign some of the variable names we used to keep notation as intuitive as possible.

It remains to check Property 4d. Let $\Psi\colon\mathbb{C}^n\to\mathbb{R}$ be $\mathcal{C}^\infty$-smooth and have the following properties:

- $0\le\Psi\le 1$ on $\mathbb{C}^n$,
- $\Psi\equiv 1$ on an open set containing $\overline{B^{(n)}(0,5\delta/8)}$,
- $\operatorname{supp}(\Psi)=\overline{\{x\in\mathbb{C}^n\colon\Psi(x)\neq 0\}}\subseteq B^{(n)}(0,6\delta/8)$.

The existence of such a Ψ is clear from real analysis (it can be constructed using mollifiers). We define

$$\chi\colon\mathbb{C}^n\times b\Omega\times(0,\tau')\to[0,1]$$

by

$$(z,\zeta,\tau)\mapsto\Psi(z-\zeta).$$

We'll show that χ is as desired. We check Property 4(d)i; while doing so we assume $\zeta\in b\Omega$ and $\tau\in(0,\tau')$ to be fixed. Smoothness of $\chi(\cdot,\zeta,\tau)$ is obvious, since Ψ is smooth. We have to prove that $\chi(\cdot,\zeta,\tau)$ is $\equiv 1$ in a neighborhood of $\overline{A_\zeta(4\widetilde{\tau}+\tau)\setminus B_\zeta(4\widetilde{\tau}+\tau)}$ and $\equiv 0$ in a neighborhood of $\overline{B_\zeta(4\widetilde{\tau}+\tau)\setminus A_\zeta(4\widetilde{\tau}+\tau)}$. By choice of Ψ it suffices to show that $\overline{A_\zeta(4\widetilde{\tau}+\tau)\setminus B_\zeta(4\widetilde{\tau}+\tau)}\subseteq\overline{B^{(n)}(\zeta,5\delta/8)}$ and $\overline{B_\zeta(4\widetilde{\tau}+\tau)\setminus A_\zeta(4\widetilde{\tau}+\tau)}\subseteq\mathbb{C}^n\setminus B^{(n)}(\zeta,6\delta/8)$. We calculate:

$$\begin{aligned}\overline{A_\zeta(4\widetilde{\tau}+\tau)\setminus B_\zeta(4\widetilde{\tau}+\tau)}&\subseteq\overline{(A_\zeta\setminus B_\zeta)\,(4\widetilde{\tau}+\tau)}\\&\subseteq\overline{(B^{(n)}(\zeta,\delta/2))\,(4\widetilde{\tau}+\tau)}\\&=\overline{B^{(n)}(\zeta,\delta/2+4\widetilde{\tau}+\tau)}\\&\subseteq\overline{B^{(n)}(\zeta,\delta/2+5\widetilde{\tau})}\\&\subseteq\overline{B^{(n)}(\zeta,\delta/2+5\delta/64)}\\&\subseteq\overline{B^{(n)}(\zeta,5\delta/8)},\end{aligned}$$

and

$$\begin{aligned}\overline{B_\zeta(4\widetilde{\tau}+\tau)\setminus A_\zeta(4\widetilde{\tau}+\tau)}&\subseteq\overline{(B_\zeta\setminus A_\zeta)\,(4\widetilde{\tau}+\tau)}\\&\subseteq\overline{\{z\in\mathbb{C}^n\colon\|z-\zeta\|>\delta\}(4\widetilde{\tau}+\tau)}\\&\subseteq\overline{\{z\in\mathbb{C}^n\colon\|z-\zeta\|>\delta\}(5\delta/64)}\\&\subseteq\overline{\mathbb{C}^n\setminus B^{(n)}(\zeta,6\delta/8)}\end{aligned}$$

$$= \mathbb{C}^n \setminus B^{(n)}(\zeta, 6\delta/8).$$

Property 4(d)ii is clear. We check Property 4(d)iii: Since Ψ is smooth and has compact support we have

$$\left|\overline{\partial}\Psi\right|_{\mathbb{C}^n} < \infty.$$

Set $K' := \left|\overline{\partial}\Psi\right|_{\mathbb{C}^n} + 1 \in \mathbb{R}_{>0}$. Then we have for $\zeta \in b\Omega, \tau \in (0, \tau')$:

$$\begin{aligned}\left|\overline{\partial}(\chi(\cdot,\zeta,\tau))\right|_{C_\zeta(4\widetilde{\tau}+\tau)} &\leq \left|\overline{\partial}\Psi\right|_{\mathbb{C}^n} \\ &< K'.\end{aligned}$$

We proceed to checking Property 4(d)iv: Fix $\tau \in (0, \tau')$ and $j, k \in \{1, \dots, n\}$. Let $c\colon \{(z,\zeta) \in \mathbb{C}^n \times b\Omega\colon z \in C_\zeta(4\widetilde{\tau}+\tau)\} \to \mathbb{C}^n$ be continuous and let $c(\cdot,\zeta)$ be holomorphic and bounded on $C_\zeta(4\widetilde{\tau}+\tau)$ for all ζ. We define the map

$$\Phi_{j,k,\tau}(c)\colon \{(p,\zeta) \in \mathbb{C}^n \times b\Omega\colon p \in \overline{D_\zeta(4\widetilde{\tau}+3\tau/4)}\} \to \mathbb{C}$$

by

$$(p,\zeta) \mapsto \begin{cases} c(p,\zeta)_j \cdot \frac{\partial(\chi(\cdot,\zeta,\tau))}{\partial \bar{z}_k}(p) & \text{if } p \in C_\zeta(4\widetilde{\tau}+\tau), \\ 0 & \text{otherwise.}\end{cases}$$

We have to check that $\Phi_{j,k,\tau}(c)$ is continuous. $\Phi_{j,k,\tau}(c)$ is defined on

$$\begin{aligned}\{(p,\zeta) \in \mathbb{C}^n \times b\Omega\colon p \in \overline{D_\zeta(4\widetilde{\tau}+3\tau/4)}\} &= \overline{\overline{\Omega}(4\widetilde{\tau}+3\tau/4)} \times b\Omega \\ &= \overline{\Omega(4\widetilde{\tau}+3\tau/4)} \times b\Omega,\end{aligned}$$

a first-countable space, so we can check continuity using the sequence criterion for continuity. Let $(p,\zeta) \in \overline{\Omega(4\widetilde{\tau}+3\tau/4)} \times b\Omega$ and let $((p_m,\zeta_m))_{m\in\mathbb{Z}_{\geq 0}}$ be a sequence in $\overline{\Omega(4\widetilde{\tau}+3\tau/4)} \times b\Omega$ converging to (p,ζ). We have to show that $\big(\Phi_{j,k,\tau}(c)\big)(p_m,\zeta_m) \xrightarrow{m\to\infty} \big(\Phi_{j,k,\tau}(c)\big)(p,\zeta)$. We are going to consider the following cases separately:

- $\frac{5}{8}\delta \leq \|p-\zeta\| \leq \frac{3}{4}\delta$,
- $\|p-\zeta\| > \frac{3}{4}\delta$,
- $\|p-\zeta\| < \frac{5}{8}\delta$.

Assume first that $5\delta/8 \leq \|p-\zeta\| \leq 3\delta/4$. Then there exists an $M \in \mathbb{Z}_{\geq 0}$ such that

$$\left(\frac{1}{2}+\frac{6}{64}\right)\cdot\delta < \|p_m-\zeta_m\| < \frac{7}{8}\delta \text{ for all } m \in \mathbb{Z} \text{ with } m \geq M.$$

Using this, an easy calculation shows that

$$\begin{aligned}&p \in C_\zeta(4\widetilde{\tau}+\tau), \\ &p_m \in C_{\zeta_m}(4\widetilde{\tau}+\tau) \text{ for all } m \in \mathbb{Z} \text{ with } m \geq M,\end{aligned}$$

which in turn implies that

$$\text{(F2)}\qquad \begin{aligned} \big(\Phi_{j,k,\tau}(c)\big)(p,\zeta) &= c(p,\zeta)_j \cdot \frac{\partial(\chi(\cdot,\zeta,\tau))}{\partial \bar{z}_k}(p), \\ \big(\Phi_{j,k,\tau}(c)\big)(p_m,\zeta_m) &= c(p_m,\zeta_m)_j \cdot \frac{\partial(\chi(\cdot,\zeta_m,\tau))}{\partial \bar{z}_k}(p_m) \text{ for all } m \in \mathbb{Z}_{\geq M}. \end{aligned}$$

The map

$$\phi_{k,\tau} \colon \mathbb{C}^n \times b\Omega \to \mathbb{C}, (p', \zeta') \mapsto \frac{\partial(\chi(\cdot,\zeta',\tau))}{\partial \bar{z}_k}(p')$$

is continuous, since χ trivially extends to a smooth map $\widetilde{\chi} \colon \mathbb{C}^n \times \mathbb{C}^n \times (0,\tau') \to \mathbb{C}$. Combining this with continuity of c and (F2) we obtain

$$\big(\Phi_{j,k,\tau}(c)\big)(p_m,\zeta_m) \xrightarrow{m\to\infty} \big(\Phi_{j,k,\tau}(c)\big)(p,\zeta),$$

as desired.

Assume now that $\|p-\zeta\| > 3\delta/4$. Then there exists an $M' \in \mathbb{Z}_{\geq 0}$ such that

$$\|p_m - \zeta_m\| > \frac{3}{4}\delta \text{ for all } m \in \mathbb{Z} \text{ with } m \geq M'.$$

The map

$$\frac{\partial(\chi(\cdot,\zeta,\tau))}{\partial \bar{z}_k} \colon \mathbb{C}^n \to \mathbb{C}$$

is $\equiv 0$ on $\mathbb{C}^n \setminus \overline{B^{(n)}(\zeta, 6\delta/8)}$ by choice of χ, so, since $\|p-\zeta\| > 3\delta/4$, we have

$$\big(\Phi_{j,k,\tau}(c)\big)(p,\zeta) = 0,$$

(independently from whether $p \in C_\zeta(4\widetilde{\tau}+\tau)$ or $p \notin C_\zeta(4\widetilde{\tau}+\tau)$). Analogously we get

$$\big(\Phi_{j,k,\tau}(c)\big)(p_m,\zeta_m) = 0 \text{ for all } m \in \mathbb{Z} \text{ with } m \geq M',$$

so in this case we trivially have

$$\big(\Phi_{j,k,\tau}(c)\big)(p_m,\zeta_m) \xrightarrow{m\to\infty} \big(\Phi_{j,k,\tau}(c)\big)(p,\zeta).$$

The last case, where $\|p-\zeta\| < \frac{5}{8}\delta$, can be handled analogously to the case we just considered.

Property 4(d)v can be checked analogously to Property 4(d)iv, namely by picking (z,ζ) in the domain of Φ_1 (resp. Φ_2) and considering the cases $\|z-\zeta\| > 3\delta/4$ and $\|z-\zeta\| \leq 3\delta/4$ (resp. $\|z-\zeta\| < 5\delta/8$ and $\|z-\zeta\| \geq 5\delta/8$) separately.

We have shown that $(\{(A_\zeta, B_\zeta)\}_{\zeta \in b\Omega}, \widetilde{\tau})$ is pleasant, so we can finish the proof by applying Theorem 3.6. $\square$

CHAPTER 5

Final Remarks

In this chapter we give a quick overview over our results and make suggestions for possible future research.

1. Conclusion

We compiled a list of assumptions that allowed us to deduce that certain biholomorphic maps close to the identity depending continuously on a parameter admit compositional splittings such that the maps obtained from said splittings depend continuously on the parameter and are close to the identity, injective and holomorphic. This was achieved by adapting the proof of a splitting lemma for biholomorphic maps by F. Forstnerič to our situation and ensuring continuous dependence on the parameter along the way. Our result can be seen as a parameter version of the original result.

We later applied that result to the case where we were presented with a family $\{\gamma_\zeta\}_{\zeta \in b\Omega}$ (where Ω is a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^2$-boundary) of injective holomorphic maps close to the identity, depending continuously on ζ and defined on open neighborhoods of sets of the form $C_\zeta = A_\zeta \cap B_\zeta$, where

$$\begin{aligned} A_\zeta &= \overline{\Omega} \cap \overline{B^{(n)}(\zeta, \delta)}, \\ B_\zeta &= \left(\overline{\Omega} \cap \overline{B^{(n)}(\zeta, \delta) \setminus B^{(n)}(\zeta, \delta/2)}\right) \cup \left(\overline{\Omega} \setminus A_\zeta\right), \end{aligned}$$

for a fixed small $\delta > 0$, to find injective holomorphic maps α_ζ and β_ζ defined (and close to the identity) on open neighborhoods of A_ζ resp. B_ζ, such that α_ζ and β_ζ depend continuously on ζ and satisfy

$$\gamma_\zeta = \beta_\zeta \circ \alpha_\zeta^{-1}$$

on a neighborhood of C_ζ.

Thus we achieved the goal presented in the introduction.

2. Future Work

We list some suggestions for future work and research:

- The number ϵ_η in Theorem 4.2 does not only depend on η, but also on δ and $\widetilde{\tau}$. It is, however, not unlikely that ϵ_η can be chosen in a way that it does *not* depend on $\widetilde{\tau}$. A result in that direction could be obtained by looking at our proofs in Chapter 3 and finding out how all the constants (whose existence we proved) depend on $\widetilde{\tau}$. This should be easy to do, since we gave all the constants explicitely.

- We have worked in $\mathbb{C}^n$ the entire time, since the applications only required results in $\mathbb{C}^n$. The original result by F. Forstnerič, however, was formulated for complex manifolds and recently has been generalized to hold for complex spaces by the same author ([**5**, Theorem 3.2 on p. 13]). It is therefor natural to ask whether our results can be generalized to hold for families of maps defined on certain open subsets of complex manifolds (resp. complex spaces). This can probably be done by comparing the proof(s) given here with the proof given by F. Forstnerič and adding some assumptions in Definition 3.4. A generalization of our result to complex manifolds could probably be used to prove a theorem about embeddings of Riemann surfaces.

- The original result due to F. Forstnerič also mentioned a holomorphic foliation $\mathcal{F}$ on the complex manifold X. One might therefor ask, whether a family of $\mathcal{F}$-maps depending continuously on a parameter will admit splittings by $\mathcal{F}$-maps depending continuously on the parameter.

- For some applications it would be nice to have smooth dependence of α_ζ and β_ζ on the parameter ζ, provided γ_ζ depends smoothly on ζ. This is probably a lot harder than the first two suggestions we mentioned. We constructed our maps $\alpha = \alpha(z, \zeta)$ and $\beta = \beta(z, \zeta)$ as uniform limits of continuous maps which allowed us to deduce continuity. A uniform limit of smooth functions, however, is not necessarily smooth, so we can *not* deduce smooth dependence on the parameter by simply rewriting resp. adapting our proof. If it is indeed possible to ensure that the maps obtained from our splitting depend smoothly on the parameter, then it will probably require a lot of additional work.

- As mentioned earlier, Theorem 2.24 only allows us to solve the $\overline{\partial}$-equation with the same constant, but does *not* give us that the solution operators *themselves* depend continuously on the domain in some sense. One might try to look at the proof of Theorem 2.22 and deduce such a continuous dependence. Assuming that this works out, one can show that Theorem 3.6 still holds if we replace 4c in Definition 3.4 by the assumption that (rougly speaking) the domains D_ζ are the closures of bounded strictly pseudoconvex domains Ω_ζ with $\mathcal{C}^2$-boundaries in $\mathbb{C}^n$, which admit $\mathcal{C}^2$ defining functions ρ_ζ, such that ρ_ζ varies continuously with ζ in $\mathcal{C}^2$-norm (and probably some compactness-assumption on the parameter space $\mathcal{P}$ to ensure the existence of the constant C). This will take away from the generality of Theorem 3.6, but will make it *a lot* easier to apply.

- The original result due to F. Forstnerič is an important result with many applications. One could try to use the results from this thesis to obtain parameter versions of said applications.

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